Exercise Sheet 13 for Infinite Graph Theory, WS 2019/20
(to be discussed on 27. January 2020)

1. Let $\triangleleft$ be X well-quasi order. Given an infinite cardinal κ we say that an element $x \in X$ is *κ-embeddable in X* if there are at least κ many elements $x' \in X$ such that $x \triangleleft x'$.
   Show that for any infinite cardinal κ the number of elements of X which are not κ-embeddable with respect to $\triangleleft$ in X is less than κ.

2. Show that for the minor ubiquity conjecture for countable graphs, we can always assume that the host graph does not have any infinitely dominated ends.

3. Halin's theorem 8.2.5(i) says that the ray is $\subseteq$-uniquitous, and on Sheet 8 Q7 we have seen that the infinite comb is not $\subseteq$-uniquitous. Show that every subdivided comb with finitely many leaves is $\subseteq$-uniquitous.

4. Show that any connected graph with infinitely many ends has infinitely many edge-disjoint double rays.

5. Let G be a graph with a thin end ω. Show that for any infinite collection of rays $\mathbb{R} \subseteq \omega$ there is an infinite subcollection $\mathcal{R}' \subseteq \mathcal{R}$ such that any two members of $\mathcal{R}'$ intersect in infinitely many vertices.

Optional:

6.$^{++}$Can you characterise which trees are $\subseteq$-uniquitous? This might be hard; can you find interesting examples of trees which are $\subseteq$-uniquitous? E.g. an infinite star? An infinite star of rays? Can you find locally finite $\subseteq$-uniquitous tree with infinitely many leaves?

Hinweise

1. Similar to the corollary on ω-embeddabilty in the lecture.
2. $TK^{\aleph_0}$.
3. Strong linking lemma.
4.
5. Ramsey
6. $^{++}$