Exercise Sheet 11 for Infinite Graph Theory, WS 2019/20
(to be discussed on 13. January 2020)

1.$^-$ Show that each of the following operations performed on the Rado graph R leaves a graph isomorphic to R:
   (i) taking the complement, i.e. changing all edges into non-edges and vice versa;
   (ii) deleting finitely many vertices;
   (iii) changing finitely many edges into non-edges or vice versa;
   (iv) changing all the edges between a finite vertex set $X \subseteq V(R)$ and its complement $V(R) \setminus X$ into non-edges, and vice versa.

2.$^-$ Prove directly that the Rado graph is homogeneous.

3. Show that there is no universal locally finite connected graph for the subgraph relation.

4. Recall that subgraphs $H_1, H_2, \ldots$ of a graph G are said to *decompose* G if their edge sets form a partition of $E(G)$. Show that the Rado graph can be decomposed into any given countable set of countable locally finite graphs, as long as each of them contains at least one edge.

5. Given a bijection f between $\mathbb{N}$ and $[\mathbb{N}]^{<\omega}$, let G_f be the graph on $\mathbb{N}$ in which $u, v \in \mathbb{N}$ are adjacent if $u \in f(v)$ or vice versa. Prove that all such graphs G_f are isomorphic.

6. Prove Theorems 1 and 2 from the lectures on Fraissé's Theorem.

Optional:

7.$^+$ Construct, under the Continuum Hypothesis, a homogeneous graph of size $\aleph_1$ which is universal for all graphs of size at most $\aleph_1$.

Hinweise

1.$^-$ Property $(*)$.

2.$^-$ Back-and-forth.

3. Suppose there is a universal graph G. Construct a locally finite connected graph H whose vertex degrees 'grow too fast' for any embedding of H in G.

4. Find the partition inductively, deleting the edge set of one graph at a time and showing that what remains is still isomorphic to R. ensure that, once all the required edge sets have been deleted, there is no edge left?

5. R.

6. Back-and-forth. Use local finiteness to express your structure as the union of a countable chain of finite substructures. Prove first Theorem 2, then Theorem 1.

7.$^+$ Consider a suitable extension of property $(*)$. The Continuum Hypothesis is needed for showing that there are only $\aleph_1$ many functions $f\colon \mathbb{N} \to \omega_1$.