Exercise Sheet 10 for Infinite Graph Theory, WS 2019/20
(to be discussed on 6. January 2020)

1. Construct an example of a small limit of large waves. Can you find a locally finite one?

2.+ Prove Theorem 8.4.2 for trees.

3. Show that the marriage theorem in its naive cardinality version holds for locally finite graphs but fails for arbitrary countable graphs with infinite degrees.

The next exercise concerns Steffens Theorem 8.4.11 and the notion of augmenting paths/rays introduced just before.

4. Let G be a countable graph in which for every partial matching there is an augmenting path.
   (i) Find an example of G and a sequence $M_0, M_1, \dots$ of partial matchings, each obtained from the previous as its symmetric difference with the edge set of an augmenting path, so that for every edge e of G we have $e \in M_{n+1} \setminus M_n$ for infinitely many n.
   (ii) Show that for every partial matching M there exists a sequence as in (i) such that $\bigcup_m \bigcap_{n>m} M_n$ is the edge set of a 1-factor.

5. Find an uncountable graph in which every partial matching admits an augmenting path (finite or infinite) but which has no 1-factor.

6.+ Prove Pym's theorem (8.4.7).

7.+ (Holiday Special – freiwillig)
A group of infinitely many prisoners are offered the following game. On New Year's Eve, each of the prisoners will receive a label on his forehead showing some real number. Every prisoner can see the other inmates' labels but not his own. They know each other and can tell each other apart. The moment they hear the New Year Canon, which by tradition is fired once to announce an amnesty at the beginning of the new year, they each have to shout a real number. If all but finitely many prisoners shout the number on their own forehead, then all prisoners are freed.

The prisoners are allowed to get together beforehand to agree a strategy. Can you advise them? You may assume the axiom of choice. You may also assume that real numbers can be written on foreheads and shouted out in an instance.

Hinweise

1.$^{-}$ Pick $a \in A$, and construct a sequence of waves $\mathcal{W}_1, \mathcal{W}_2, \ldots$ that each contain the trivial path $\{a\}$. Define the edges at a so that a is in the boundary of every $\mathcal{W}_n$, but not in the boundary of the limit wave.

2.$^{+}$ The general problem reduces to Lemma 8.4.3, just as in the countable case. Prove the lemma for forests.

3. For the locally finite case use compactness / infinity lemma.

4. By Theorem 8.4.11, G has a 1-factor. Can you use it for a positive solution for (ii)?

5. To ensure that every partial matching can be augmented, give your graph lots of edges. How can you nevertheless prevent a 1-factor?

6.$^{+}$ Starting with $\mathcal{P}$, recursively define path systems $\mathcal{P}_\alpha$ that link A to more and more of B. In the recursion step, pick an uncovered vertex $b \in B$ and follow the path $Q \in \mathcal{Q}$ containing it back until it hits $\mathcal{P}_\alpha$, say in $P = a \ldots b'$. You could then re-route P to follow Q to b from there, but this would leave b' uncovered. Still, could it be that these changes produce an increase of the covered part of B at limit steps? To prove that it does, can you define an 'index' parameter that grows (or decreases) with every step but cannot do so indefinitely? Alternatively, prove and apply a suitable infinite version of the stable marriage theorem (2.1.4).

7.$^{+}$