Complex Haar spaces generated by shifts applied to a single function

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Abstract. Some of the known Haar spaces are generated by shifts of a single function G. There are examples of two types. In one case the spaces generated are real spaces defined on compact intervals, in the other case the generated spaces are also Haar spaces on compact subsets of the complex plane $\mathbb{C}$. Under the assumption that G is analytic we are able to characterize those functions G which generate Haar spaces in the plane $\mathbb{C}$.

§1. Background

We denote by $\mathbb{N}$ the set of all positive integers, by $\mathbb{Z}$ the integers, by $\mathbb{R}$ the real and by $\mathbb{C}$ the complex numbers. The letter $\mathbb{K}$ stands for either $\mathbb{R}$ or $\mathbb{C}$ and Π_n denotes the set of all polynomials with degree at most n, where, in general, complex coefficients are permitted. We will distinguish between the equality sign $=$ and the sign $:=$ (possibly also $=:$) where the latter stands for a defining relation. What is defined appears on the side where the colon : is. Let D be any compact Hausdorff space and $X := \mathrm{C}(D)$ the set of all continuous functions defined on D with values in $\mathbb{K}$. With the norm $||f|| := \max_{t\in D} |f(t)|$ for all $f \in X$, usually called the uniform norm, the space X becomes a Banach space over $\mathbb{K}$, or since functions can even be multiplied pointwise, a Banach algebra. The main underlying theorem is the following.

Theorem 1. *For a fixed $n \in \mathbb{N}$ let $H \subset X$ be an n-dimensional subspace of X. The following four statements are equivalent:*

i. *For any selection of n pairwise distinct points $t_j \in D$ and any set of n numbers $\eta_j \in \mathbb{K}$ the interpolation problem*

$$h(t_j) = \eta_j, \quad j = 1, 2, \ldots, n$$

has a unique solution $h \in H$.

Approximation Theory X
Charles K. Chui, Larry L. Schumaker, and Joachim Stoeckler (eds.), pp. 1–14.

ISBN 0-8265-xxxx-x.

ii. *Any $h \in H\backslash\{0\}$ has at most $n-1$ zeros in D.*

iii. *Let $H := \langle h_1, h_2, \dots, h_n \rangle$, i.e. H is the linear hull of the linearly independent functions $h_j \in X$, $j = 1, 2, \dots, n$. Then the $(n \times n)$ matrix*

$$\mathbf{M} := (h_j(t_k)), \quad j, k = 1, 2, \dots, n$$

is non-singular for any choice of pairwise distinct points $t_j \in D$, $j = 1, 2, \dots, n$.

iv. *Any $f \in X$ possesses a unique best approximation $\hat{h} \in H$, i.e. $||f - \hat{h}|| < ||f - h||$ for all $h \in H\backslash\{\hat{h}\}$.*

Proof: Meinardus, [12, 1967, p. 16–17]. □

Definition 2. A finite dimensional subspace H of X with dimension n is called a Haar space, if it possesses one of the properties given in Theorem 1.

Unfortunately, Haar spaces do not generally exist. This is governed by the following two theorems.

Theorem 3. *Let $\mathbb{K} = \mathbb{R}$. An n-dimensional subspace $H \subset X$ with $n \geq 2$ can only be a Haar space if D is topologically equivalent to a closed subset of the unit circle $C := \{z \in \mathbb{C} : |z| = 1\}$ with at least n points.*

Proof: Mairhuber, [11, 1956], Curtis, [5, 1958]. □

This result reduces Haar spaces in the real case essentially to the following cases: D is a compact interval in $\mathbb{R}$, D is a complete circle, which reduces H to p-periodic functions on $[0, p[\subset \mathbb{R}$, D consists of finitely many points (at least n).

Theorem 4. *Let $\mathbb{K} = \mathbb{C}$ and D locally connected. An n-dimensional subspace $H \subset X$ with $n \geq 2$ can only be a Haar space if D is topologically equivalent to a closed subset of $\mathbb{C}$ and contains at least n points.*

Proof: Henderson & Ummel, [8, 1973]. □

Historically the first theorem restricting the domain of definition D in the case of complex Haar spaces $H \subset X$ to subsets of $\mathbb{C}$ was given by Schoenberg & Yang, [16, 1961]. The consequence of Theorem 3 and Theorem 4 is that Haar spaces for $D \subset \mathbb{K}^n$ with $n \geq 2$ do not in general exist.

§2. Examples

In the literature there are lists of Haar spaces which are generally not very long. Standard sources are Karlin & Studden [9, 1966] and Dunham [6, 1974]. One finds two examples where the Haar spaces are generated by shifts applied to a single function f. These examples are $f(t) := 1/t$ and $f(t) := \exp(-t^2)$. In the end of § 3 we will mention another example. The main question of this paper is whether one can characterize the functions with this property. This problem was also raised for the real case by Cheney & Light [4, 2000, p. 76]. Radial basis splines (Cheney & Light [4, 2000, Ch. 15 & 16]) evolved also from shifts applied to a single function. Let us first inspect the two mentioned examples.

Example 5. Let $f = 1/t$ for all $t \in \mathbb{C}\backslash\{0\}$ and let D be any compact set in $\mathbb{C}$ and $S := \complement D$ with respect to $\mathbb{C}$, where $\complement$ stands for the complement. Let $s_j \in S$, $j \in \mathbb{N}$ any sequence of pairwise distinct points and $h_j(t) := f(t - s_j)$, $j \in \mathbb{N}$. Then for all n, the spaces $H_n := \langle h_1, h_2, \ldots, h_n\rangle$ are Haar spaces with $H_n \subset H_{n+1}$. This is easy to see: A typical element of H_n is of the form

$$\eta_n := \sum_{j=1}^{n} a_j h_j, \quad a_j \in \mathbb{C}.$$

This expression can be given the form

$$\eta_n(t) = \sum_{j=1}^{n} a_j h_j(t) = \frac{1}{\prod_{j=1}^{n}(t - s_j)} \sum_{j=1}^{n} a_j \prod_{k \neq j}(t - s_k) =: \frac{p(t)}{q(t)},$$

where $p \in \Pi_{n-1}, q \in \Pi_n$. Since q has no zeros in D, η_n is well defined and η_n has at most $n-1$ zeros in D, since p has at most $n-1$ zeros in $\mathbb{C}$, provided p is non-trivial.

In this example $f(t) := 1/t$ universally generates Haar spaces in the real and in the complex case.

Example 6. Let $f(t) := \exp(-t^2)$, $t \in \mathbb{K}$. We consider spaces spanned by $h_j(t) := f(t - s_j)$ with arbitrary but pairwise distinct shifts $s_j \in \mathbb{K}$, $j \in \mathbb{N}$. We have $h_j(t) = \exp(-s_j^2)\exp(-t^2)\exp(2s_j t)$. Thus, for all $n \in \mathbb{N}$ we have

$$H_n := \langle h_1, h_2, \ldots, h_n\rangle = \exp(-t^2)\langle\exp(2s_1 t), \exp(2s_2 t), \ldots, \exp(2s_n t)\rangle.$$

Since e^z has no zeros for all $z \in \mathbb{C}$, the problem is reduced to the investigation of the space

$$\tilde{H}_n := \langle\exp(s_1 t), \exp(s_2 t), \ldots, \exp(s_n t)\rangle.$$

If $\mathbb{K} = \mathbb{R}$ the space $\tilde{H}_n$ belongs in the catalogue of well known Haar spaces, Karlin & Studden, [9, 1966, Example 1, p. 9-10]. If, however, $\mathbb{K} = \mathbb{C}$ let us consider $\tilde{H}_2$. According to Theorem 1, part iii, $\tilde{H}_2$ is a Haar space if and only if $\det \mathbf{M} \neq 0$, where $\mathbf{M} := (\exp(s_j t_k))$, $j, k = 1, 2$ for arbitrary $t_1 \neq t_2$. Now, $\det \mathbf{M} = \exp(s_1 t_1 + s_2 t_2) - \exp(s_1 t_2 + s_2 t_1)$. This expression is zero if and only if $s_1 t_1 + s_2 t_2 = s_1 t_2 + s_2 t_1 + 2k\pi\mathrm{i}$ for some $k \in \mathbb{Z}$. This is equivalent to $(t_1 - t_2)(s_1 - s_2) = 2k\pi\mathrm{i}$. Now it is easy to see, that for $k \neq 0$ there are solutions with $t_1 \neq t_2$ for any given $s_1 \neq s_2$. One example with $k \neq 0$ is $s_1 = \pi\mathrm{i}/2, s_2 = -\pi\mathrm{i}/2$; $t_1 = k, t_2 = -k$. In the complex case the spaces $\tilde{H}_n$ are no longer Haar spaces for $n \geq 2$.

In this example $f(t) := \exp(-t^2)$ does not universally generate Haar spaces in the real and in the complex case.

§3. The complex analytic case

We will denote by $\mathbb{D}$ the open, unit disk in $\mathbb{C}$ and correspondingly $\overline{\mathbb{D}}$ will denote the closed, unit disk.

Definition 7. Let $n \in \mathbb{N}$ be a fixed natural number. A function G defined on $\mathbb{C}\backslash\{0\}$ with values in $\mathbb{C}$ will be called an n-dimensional Haar space generator, if for each set of n pairwise distinct points $s_1, s_2, \ldots, s_n \in \mathbb{C}\backslash\overline{\mathbb{D}}$, the functions h_j defined by $h_j(z) := G(z - s_j)$, $j = 1, 2, \ldots, n$ for $z \in \overline{\mathbb{D}}$ span an n-dimensional Haar space.

Example 8. Let $G(z) := z^{m-1}$ with $m \geq 1$ fixed. Then G is an m-dimensional Haar space generator but not an $(m+1)$-dimensional Haar space generator. We leave open the case whether it is an ℓ-dimensional Haar space generator for $2 \leq \ell < m$.

Definition 9. A function G defined on $\mathbb{C}\backslash\{0\}$ with values in $\mathbb{C}$ will be called a universal Haar space generator, if for each $n \in \mathbb{N}$, it is an n-dimensional Haar space generator. A universal Haar space generator will be abbreviated by UHG.

Theorem 10. *Let G be analytic on $\mathbb{C}\backslash\{0\}$. Then G is a UHG if and only if G is of the form*

$$G(z) := \frac{e^{az+b}}{z}, \quad a \in \mathbb{C},\ b \in \mathbb{C}. \tag{1}$$

Proof: a) Sufficiency of the condition: It is essentially the same proof as in Example 5. Fix a and $b \in \mathbb{C}$. Then the n functions generated by shifts from G have the form

$$h_j(z) := A\frac{e^{a(z-s_j)}}{z - s_j}, \quad 1 \leq j \leq n, \quad A \neq 0.$$

They are linearly independent since the poles at s_j cannot be removed by non-trivial linear combinations $\eta_n := \sum_{j=1}^{n} \mu_j h_j$. Now, $\eta_n(z) = \sum_{j=1}^{n} \mu_j h_j(z) = e^{az}\frac{p(z)}{q(z)}$ where p is a polynomial of degree at most $n-1$ and q a polynomial of degree n, where the zeros of q are not in $\overline{\mathbb{D}}$. Therefore, each non-trivial η_n has at most $n-1$ zeros in $\mathbb{C}$ and hence in $\overline{\mathbb{D}}$. We conclude that G is a UHG.

b) Necessity of the condition: Suppose G is an analytic UHG. The proof that G is of the form (1) is separated into several Lemmata.

Lemma 11. *G does not vanish on $\mathbb{C}\backslash\{0\}$.*

Proof: Let $n = 1$, and suppose that G vanishes at $z_0 \neq 0$. If $|z_0| > 1$, then $\tilde{G}(z) := G(z + z_0) = G(z - (-z_0))$ vanishes at the origin where $-z_0 \notin \overline{\mathbb{D}}$. If $z_0 \in \overline{\mathbb{D}}\backslash\{0\}$, put $s = -\frac{2+|z_0|}{2|z_0|}z_0$. Then $s \notin \overline{\mathbb{D}}$ and $\tilde{G}(z) := G(z - s)$ vanishes at $z := \frac{-2+|z_0|}{2}\frac{z_0}{|z_0|}$ which is in $\mathbb{D}$. Both cases are hence excluded by the hypothesis that G is a UHG, and in particular a 1-dimensional Haar space generator. □

Lemma 12. *G has the form*

$$G(z) := z^m e^{\phi(z)}, \quad z \in \mathbb{C}, \quad m \in \mathbb{Z}, \quad \textit{where } \phi \textit{ is an entire function.} \tag{2}$$

Proof: Define the function

$$F(z,t,s) := \frac{G(z-t)}{G(z-s)}, \quad |z| \le 1, |t| > 1, |s| > 1, s \ne t. \tag{3}$$

This definition is permissible since by Lemma 11, G has no zeros in $\mathbb{C}\setminus\{0\}$. We are studying the number of solutions of $F(z,t,s) = \mu$ with a constant $\mu \in \mathbb{C}\setminus\{0\}$ since this is equivalent to studying the number of zeros of the linear combination $G(z-t) - \mu G(z-s)$ of two shifts, where $t \ne s$ and $t, s \notin \overline{\mathbb{D}}$. The function G has by definition an isolated singularity at the origin. There are only three possibilities: (a) This singularity is removable, (b) this singularity is a pole, or (c) this singularity is an essential singularity. We shall now show, that the case (c) is not possible. Suppose that the origin is an essential isolated singularity for G. Choose $t > 1$ and $s > t + 10$ on the positive real axis such that $G(t-s)$ is well defined and, according to Lemma 11, $G(t-s) \ne 0$. Then, by Picard's Theorem, (Ahlfors [1, 1966, p. 297]) there is a $\mu \in \mathbb{C}$ such that $F(z,t,s) = \mu$ admits infinitely many solutions in z in every neighbourhood of $z = t$. Furthermore, there is a cone $\{z : |\arg(z-t) - \beta| < \frac{\pi}{4}\}$ ending at t which contains infinitely many zeros of $F(z,t,s) - \mu$. Put $t_1 := -(1+\epsilon)e^{i\beta}$, $z_1 := z - t + t_1$ and $s_1 := s - t + t_1$. We may choose $\epsilon > 0$ sufficiently small such that $F(z_1, t_1, s_1) - \mu$ vanishes at several points in $\mathbb{D}$. Observe that t_1 and s_1 are outside $\overline{\mathbb{D}}$. Hence, G is not a 2-dimensional Haar space generator. Thus, G has an isolated singularity at the origin, which is either removable or a pole, and in addition G has no zeros in $\mathbb{C}\setminus\{0\}$. In other words, we have shown that G has to be of the form defined in (2). Furthermore, all F defined by (3) are meromorphic on $\mathbb{C}$ having at most one pole and $F(0,t,s) \ne 0$, $F(0,t,s) \ne \infty$. □

Lemma 13. *Let $\mu \in \mathbb{C}\setminus\{0\}$ be fixed and let F be defined as in (3). Then $F - \mu$ has at most one zero in the disk $\Delta(p) := \{z : |z-p| \le 1\}$ provided $|p| > |t| + |s|$. Furthermore, this zero is simple.*

Proof: Fix $|s| > 1, |t| > 1$, and $|p| > |t| + |s|$ and set $z_1 := z - p$, $t_1 := t - p$, $s_1 := s - p$. Then $F(z,t,s) - \mu$ contains at most one simple zero on the disk $\Delta(p)$. Indeed, $F(z,t,s) - \mu$ vanishes at a point $z^\star \in \Delta(p)$ if and only if (setting $z_1^\star := z^\star - p$)

$$F(z_1^\star, t_1, s_1) - \mu = F(z^\star - p, t - p, s - p) - \mu = 0. \tag{4}$$

We have $|z_1^\star| \le 1, |t_1| = |p - t| \ge |p| - |t| > |t| + |s| - |t| = |s| > 1$ and analogously $|s_1| = |p - s| > 1$ which implies the above statement. □

Lemma 14. *F defined in (3) is of the form*

$$F(z,t,s) = \frac{G(z-t)}{G(z-s)} = \left[\frac{z-t}{z-s}\right]^m e^{a(t,s)z^2 + b(t,s)z + c(t,s)}. \tag{5}$$

Proof: Consider the partition of $\mathbb{C}$ by the squares

$$R_{j,k} := \{(x,y) \in \mathbb{R}^2 : \frac{j}{2} \le x \le \frac{j+1}{2}, \quad \frac{k}{2} \le y \le \frac{k+1}{2}, \quad j,k \in \mathbb{Z}\} \tag{6}$$

and denote by $n(r,F)$ the number of zeros of the function $F(z,t,s)-\mu$ in the disk $\{z : |z| \leq r\}$ for a fixed $\mu \in \mathbb{C}\backslash\{0\}$. Fix an $r_0 \in \mathbb{N}$ with $r_0 > |t|+|s|$. Each closed disk $\{z : |z| \leq r\}$, $r \in \mathbb{N}$, is covered by $16r^2$ squares of the form (6). Moreover, $|n(r_0,F)| \leq M < \infty$ uniformly in μ. This follows directly from the argument principle for meromorphic functions, AHLFORS [1, 1966, p. 151]. Hence, by Lemma 13 we conclude that

$$n(r,F) = n(r_0,F)+[n(r,F)-n(r_0,F)] \leq n(r_0,F)+16r^2 = O(r^2), \text{ as } r \to \infty. \tag{7}$$

Observe that (7) holds for all $r > r_0$ and all $\mu \in \mathbb{C}$. Hence, by the first and second fundamental theorem of R. Nevanlinna (NEVANLINNA [14, 1953, p. 168/256]), we conclude that F is meromorphic on $\mathbb{C}$ of maximal order two, which implies that $\exp(\phi(z-t)-\phi(z-s))$ is an entire function of maximal order two. The same conclusion may also be obtained by Cartan's theorem (CARTAN, [2, 1928; 3, 1929]). Therefore, $\phi(z-t)-\phi(z-s)$ must be a polynomial of degree $d \leq 2$. In other words, F has to be of the form given in (5). □

For more details of this proof, see Section 4: Appendix.

Lemma 15. *G is of the form*

$$G(z) = z^m \mathrm{e}^{Az^3+Bz^2+Cz+D}, \quad m \in \mathbb{Z}, \quad A,B,C,D \in \mathbb{C}. \tag{8}$$

Proof: We evaluate the constants $a(t,s)$, $b(t,s)$, $c(t,s)$ occurring in (5) of Lemma 14. Let us define a polynomial (in z with parameters t,s) by

$$p(z;t,s) := \phi(z-t)-\phi(z-s) = a(t,s)z^2+b(t,s)z+c(t,s).$$

This form is motivated by the proof of the previous Lemma 14. The polynomial p has the simple property $p(z;s,t) = -p(z;t,s)$, which implies the relations $a(t,s) = -a(s,t)$, $b(t,s) = -b(s,t)$ and $c(t,s) = -c(s,t)$ for all t,s. If we compute the derivatives of p with respect to z,t,s we obtain

$$\begin{aligned} \phi'(z-t)-\phi'(z-s) &= 2a(t,s)z+b(t,s), \\ -\phi'(z-t) &= a_t(t,s)z^2+b_t(t,s)z+c_t(t,s), \\ \phi'(z-s) &= a_s(t,s)z^2+b_s(t,s)z+c_s(t,s). \end{aligned}$$

If we compare the negative of the first equation with the sum of the second and third equation we obtain (omitting the arguments)

$$a_t+a_s=0,\ b_t+b_s=-2a,\ c_t+c_s=-b.$$

Computing the second derivatives with respect to z,t,s we obtain

$$\begin{aligned} \phi''(z-t)-\phi''(z-s) &= 2a(t,s), \\ \phi''(z-t) &= a_{tt}(t,s)z^2+b_{tt}(t,s)z+c_{tt}(t,s), \\ -\phi''(z-s) &= a_{ss}(t,s)z^2+b_{ss}(t,s)z+c_{ss}(t,s). \end{aligned}$$

Comparing the sum of the last two equations with the first equation yields

$$a_{tt} + a_{ss} = 0,\ b_{tt} + b_{ss} = 0,\ c_{tt} + c_{ss} = 2a.$$

Combining these equations we obtain the partial differential equations:

$$a_{tt} = 0,\ a_{ss} = 0,\ b_{tt} = -2a_t,\ b_{ss} = -2a_s,\ c_{tt} = -b_t,\ c_{ss} = -b_s.$$

Taking the above mentioned symmetry into account, we find the solutions to be

$$a(t,s) = \alpha(t-s),\ b(t,s) = -\alpha(t^2-s^2) + \beta(t-s),$$
$$c(t,s) = \frac{\alpha}{3}(t^3-s^3) - \frac{\beta}{2}(t^2-s^2) + \gamma(t-s),$$

with arbitrary factors α, β, γ. In (5) put $z := 0$ and fix $s := s_0$. Then we obtain

$$F(0,-t,-s_0) = \frac{G(t)}{G(s_0)} = \left[\frac{t}{s_0}\right]^m \mathrm{e}^{-\frac{\alpha}{3}(t^3-s_0^3)-\frac{\beta}{2}(t^2-s_0^2)-\gamma(t-s_0)},$$

from whence we conclude that G is of the form (8). □

We summarize our results so far.

Proposition 16. *Let G be simultaneously an analytic one- and two-dimensional Haar space generator. Then G has to be of the form given in (8).*

Proof: This is a consequence of Lemma 11 to Lemma 15 where only Haar space generators up to dimension two were used. □

Lemma 17. *In G of (8) we have $A = 0$.*

Proof: Suppose that the analytic function G of (8) has the property that $A \neq 0$. We shall show in this case that G is not even a 2-dimensional Haar space generator. Indeed, we have $F(z,s,t) - 1 = 0$ is equivalent to

$$(z-t)^m \mathrm{e}^{A(z-t)^3+B(z-t)^2+C(z-t)+D} = (z-s)^m \mathrm{e}^{A(z-s)^3+B(z-s)^2+C(z-s)+D}$$

or for $k \in \mathbb{Z}$ to

$$A(t^2+ts+s^2) - (3Az+B)(t+s) + (3Az^2+2Bz+C) = \frac{m\log\left[\frac{z-t}{z-s}\right] + 2mk\pi\mathrm{i}}{t-s}.$$

We choose $s := s(t)$ in such a way that $A(t^2+ts+s^2) - B(t+s)$ (regarded as a quadratic polynomial in s) vanishes. This yields (as one possible choice)

$$s(t) := -\frac{At-B}{2A}\left[1 - \sqrt{1 - \frac{4At}{At-B}}\right]. \tag{9}$$

Observe that $s(t)$ is close to $e^{2\pi\mathrm{i}/3}t$ if t is very large. We have $F(z,s,t) - 1 = 0$ if and only if

$$z - \frac{3Az^2+2Bz+C}{3A(t+s(t))} + \frac{m\log\left[\frac{z-t}{z-s(t)}\right] + 2mk\pi\mathrm{i}}{3A(t^2-s^2(t))} = 0, \quad k \in \mathbb{Z}. \tag{10}$$

Define

$$f_k(z) := \frac{3Az^2+2Bz+C}{3A(t+s(t))} - \frac{m\log\left[\frac{z-t}{z-s(t)}\right]+2mk\pi\mathrm{i}}{3A(t^2-s^2(t))}, \qquad k\in\mathbb{Z}.$$

Then, for large t, we have $t+s\approx t(1+\mathrm{e}^{2\pi\mathrm{i}/3})$ and $t^2-s^2\approx t^2(1-\mathrm{e}^{-2\pi\mathrm{i}/3})$. Furthermore, $\log\left[\frac{z-t}{z-s(t)}\right]$ will be close to $-2\pi\mathrm{i}/3$. Now, choose $k_0\in\mathbb{N}$. Then for t very large, the functions $f_k, 0\le k\le k_0$, are analytic in the closed unit disk $\overline{\mathbb{D}}$ and $0<|f_k(z)|\le\frac{1}{2}$ on $\overline{\mathbb{D}}$. Applying Rouché's theorem (Ahlfors [1, 1966, p. 152]) to $f(z):=z$ and f_k we conclude that for each $k, 0\le|k|\le k_0$, the function $f-f_k$ vanishes exactly once in $\mathbb{D}$. Therefore, we obtain several z of the form (10) which belong to the unit disk. This contradicts the fact that G is a 2-dimensional and hence UHG. Therefore, A has to be zero in (8). □

Lemma 18. *In G of (8) we have $B=0$ and hence $\phi(z):=Cz+D$.*

Proof: We proceed the same way as in Lemma 17 assuming that $A=0$. Suppose to the contrary, that $B\neq 0$ in (8). Then $F(z,s,t)-1=0$ is equivalent to

$$(z-t)^m\mathrm{e}^{B(z-t)^2+C(z-t)+D} = (z-s)^m\mathrm{e}^{B(z-s)^2+C(z-s)+D}$$

or to

$$B(s+t)-(2Bz+C)+\frac{m\log\left[\frac{z-t}{z-s}\right]+2mk\pi\mathrm{i}}{t-s}=0, \qquad k\in\mathbb{Z}. \tag{11}$$

Choose $s(t):=\frac{C}{B}-t$. Then (11) is equivalent to

$$z=\frac{m\log\left[\frac{z-t}{z-s}\right]+2mk\pi\mathrm{i}}{2B(t-s)}=\frac{m\log\left[\frac{z-t}{z+t-\frac{C}{B}}\right]+2mk\pi\mathrm{i}}{2B(2t-\frac{C}{B})}. \tag{12}$$

Now, choose $k_0\in\mathbb{N}$. Then for t sufficiently large, the functions

$$f_k(z):=\frac{m\log\left[\frac{z-t}{z+t-\frac{C}{B}}\right]+2mk\pi\mathrm{i}}{2B(2t-\frac{C}{B})}, \quad 0\le|k|\le k_0,$$

are analytic in the closed unit disk $\overline{\mathbb{D}}$ and $0<|f_k(z)|\le\frac{1}{2}$ on $\overline{\mathbb{D}}$. Applying Rouché's theorem (Ahlfors [1, 1966, p. 152]) to $f(z):=z$ and f_k we conclude that for each $k, 0\le|k|\le k_0$, the function $f-f_k$ vanishes exactly once in $\mathbb{D}$. Therefore, we obtain several z of the form (12) which belong to the unit disk. This contradicts that G is a 2-dimensional and hence, a UHG. □

Lemma 19. *The exponent m of the representation (2) is $m=-1$.*

Proof: So far, we have shown that an analytic UHG has to be of the form

$$G(z)=z^m\mathrm{e}^{az+b}, \quad m\in\mathbb{Z}, \quad a,b\in\mathbb{C}. \tag{13}$$

Our aim is to show that $m=-1$. If $m\geq 0$, then for $n>m+1$, the functions

$$v_j(z):=(z-s_j)^m e^{a(z-s_j)+b},\quad 1\leq j\leq n$$

form a linearly dependent set. Therefore, m has to be a negative integer.

If $m=-2$, the linear combination

$$\begin{aligned} h_3(z) &:= e^{at-b}\frac{1}{(z-t)^2}e^{a(z-t)+b} \\ &+ e^{ae^{2\pi i/3}t-b-4\pi i/3}\frac{1}{(z-e^{-2\pi i/3}t)^2}e^{a(z-e^{-2\pi i/3}t)+b} \\ &+ e^{ae^{-2\pi i/3}t-b+4\pi i/3}\frac{1}{(z-e^{2\pi i/3}t)^2}e^{a(z-e^{2\pi i/3}t)+b} \end{aligned}$$

vanishes exactly at the points z where

$$\frac{1}{(z-t)^2}+\frac{1}{(e^{2\pi i/3}z-t)^2}+\frac{1}{(e^{-2\pi i/3}z-t)^2}=0.$$

This is the case if and only if $z^3=\frac{-t^3}{2}$. For $t:=1.1$ we have three zeros in the unit disk which shows that the case $m=-2$ is excluded.

Now let m be a negative integer, $m\leq -3$ and fix t on the real axis, $t>1$. Then the linear combination

$$h_2(z):=e^{-(b+at)}(z+t)^m e^{a(z+t)+b}+(-1)^m e^{-b+at}(z-t)^m e^{a(z-t)+b} \tag{14}$$

vanishes if and only if $(t+z)^m+(t-z)^m=0$ or, equivalently, if and only if

$$\psi(z):=\frac{t+z}{t-z}=(-1)^{1/m}=e^{\frac{-\pi i}{|m|}+\frac{2\pi ik}{|m|}},\quad 0\leq k\leq |m|-1. \tag{15}$$

Observe, that ψ is a univalent (conformal) mapping from the unit disk $\mathbb{D}$ to the disk

$$\Omega:=\left\{w:\left|w-\frac{t^2+1}{t^2-1}\right|<\frac{2t}{t^2-1}\right\}. \tag{16}$$

Fix $t:=1.05$. Then Ω contains the two points $w_{1,2}:=e^{\pm\pi i/|m|}$, $m\leq -3$. In other words, h_2 vanishes at the two points $z_1:=\psi^{-1}(e^{\pi i/|m|})$ and $z_2:=\psi^{-1}(e^{-\pi i/|m|})=-z_1$ which belong to $\mathbb{D}$. Therefore, the cases $m\leq -3$ are also excluded. The only remaining case is $m=-1$. □

Combining all Lemmata we have shown that an analytic UHG is of the form (1), which ends the proof of the main theorem. □

We can prove much more. In particular, we obtain the following striking result.

Theorem 20. *Let G be an analytic n-dimensional Haar space generator for $n = 1, 2, 3$ and 4. Then G is a UHG. This result is best possible in the sense that 4 cannot be replaced by a smaller number.*

Proof: Suppose that G is an analytic n-dimensional Haar space generator for $n = 1, 2, 3$ and $n = 4$. From $n = 1$, we conclude that G does not vanish on $\mathbb{C}\backslash\{0\}$. The next section for $n = 2$ in the above proof, shows that G is of the form (13). The exclusion of the cases $m \leq -2$ and $m = 0, 1, 2$ are based on the same reasoning as before and uses only $n = 2, 3$ and $n = 4$. The only modification we have to add is for the cases $m \geq 3$. We can use the same arguments as for the cases $m \leq -3$. The linear combination h_2 already defined in (14) vanishes if and only if $(t + z)^m + (t - z)^m = 0$ or, equivalently, if and only if

$$\psi(z) := \frac{t+z}{t-z} = (-1)^{1/m} = \mathrm{e}^{\frac{\pi \mathrm{i}}{m} + \frac{2\pi \mathrm{i} k}{m}}, \quad 0 \leq k \leq m-1.$$

Then ψ is a univalent (conformal) mapping from the unit disk $\mathbb{D}$ to the disk Ω defined in (16). Fix $t := 1.05$. Then Ω contains the two points $w_{1,2} := \mathrm{e}^{\pm\pi \mathrm{i}/m}, m \geq 3$. In other words, h_2 vanishes at the two points $z_1 := \psi^{-1}(\mathrm{e}^{\pi \mathrm{i}/m})$ and $z_2 := \psi^{-1}(\mathrm{e}^{-\pi \mathrm{i}/m}) = -z_1$ which belong to $\mathbb{D}$. In all the arguments we used only the fact that G was an analytic n-dimensional Haar space generator with $n = 1, 2, 3$ and 4.

It remains to show that this result is best possible. Consider $G(z) = z^2$. We shall see that G is an analytic n-dimensional Haar space generator with $n = 1, 2, 3$ but not with $n = 4$ (compare Example 8).

Since $G(z) \neq 0$ for $z \neq 0$ we conclude that G is a 1-dimensional Haar space generator. Next, suppose that G is not a 2-dimensional Haar space generator. Then there must be a $|t| > 1$, an $|s| > 1$ and a $\mu \in \mathbb{C}$ such that there are at least two points z_1 and z_2 with

$$k(z) := \frac{t-z}{s-z} = \pm\sqrt{\mu}. \tag{17}$$

Since $w = 0$ and $w = \infty$ are not in the closure of the image $k(\mathbb{D})$ we conclude that $k(\mathbb{D})$ is a disk which cannot contain μ and $-\mu$. Therefore, G is a 2-dimensional Haar space generator. Next, observe that any linear combination of $(z - t)^2, (z - s)^2$ and $(z - u)^2$, where t, s and u are mutually distinct, cannot have more than two zeros in $\mathbb{D}$. Hence, G is a 3-dimensional Haar space generator. Finally, any four functions $(z - t_j)^2$, $j = 1, 2, 3, 4$ are linearly dependent and hence, G is not a 4-dimensional Haar space generator. □

Theorem 20 may be given the following form.

Corollary 21. *Let G be analytic in $\mathbb{C}\backslash\{0\}$. For arbitrary pairwise distinct points $s_j \in \complement\overline{\mathbb{D}}$ and for arbitrary pairwise distinct $t_j \in \mathbb{D}$ we define the matrix*

$$\mathbf{M} := (m_{jk}) := \big(G(t_k - s_j)\big), \quad j, k = 1, 2, \ldots, n.$$

If $\mathbf{M}$ *is non singular for* $n = 1, 2, 3, 4$, *then it is non singular for all* $n \in \mathbb{N}$ *and* G *has necessarily the form given in (1). The number* $n = 4$ *cannot be replaced by a smaller number.*

Proof: Application of Theorem 1, part iii. to Theorem 20. □

If in the definition (1) of G the constant a vanishes, the above matrix $\mathbf{M}$ is a so-called Cauchy matrix, see KNUTH [10, 1969, p. 36, 473].

Corollary 22. *Let* $m \geq 4$ *and* $g(z) := z^{m-1}$, $s_j \in \complement\mathbb{D}$, $j = 1, 2, \ldots, m+1$, $s_j \neq s_k$ *for* $j \neq k$. *Define the spaces* H_j *on* $\mathbb{D}$ *by*

$$H_j :=< g(z - s_1), g(z - s_2), \ldots, g(z - s_j) >, \quad j = 1, 2, \ldots, m+1. \tag{18}$$

Then H_2 *and* H_{m+1} *are not Haar spaces.*

Proof: For H_2 we use the proof of Theorem 20 again, in which it is shown directly, that H_2 for $m \geq 4$ is not a Haar space. Since H_{m+1} has (at most) dimension m, it cannot be a Haar space. □

It should be remarked that in the situation of Corollary 22 the spaces $H_1 :=< (z - s_1)^m >$ and $H_m (= \Pi_{m-1})$ are always Haar spaces.

Since the sufficiency proof of Theorem 10 does not depend on any specific domain of definition, we have the following corollary.

Corollary 23. *Let* $D \subset \mathbb{C}$ *be any compact set,* $S := \complement D$, G *defined as in (1), and* $n \in \mathbb{N}$. *Then for any selection of* n *mutually disjoint points* $s_j \in S$ *the space* $H_n :=< h_1, h_2, \ldots, h_n >$ *is a Haar space with domain of definition* D, *where* $h_j(z) := G(z - s_j)$, *provided* D *contains at least* n *points.*

Proof: A direct repitition of part a) of the proof of Theoem 10. □

By Theorem 10 we also can identify many non-Haar spaces. One set of examples is produced by $g(z) := \frac{1}{z^m}$, $m \geq 2$ and the corresponding spaces generated by shifts. For $m = 2$ this was shown by RAHMAN & RUSCHEWEYH [15, 1983]. In the real case, the spaces $H_n :=< g(z - s_1), g(z - s_2), \ldots, g(z - s_n) >$ defined on $I := [-1, 1]$ are Haar spaces for all $m \geq 2$, provided, $|s_j| > 1$ and the points $s_1, s_2, \ldots, s_n$ are mutually distinct. Actually, in the mentioned literature and in the forthcoming book by MEINARDUS & WALZ [13, $\geq$ 2002, p. 104] the spaces spanned by

$$v_j(x) := \left(\frac{1}{1 - x_j x}\right)^m, \quad 0 < |x_j| < 1, \quad x \in [-1, 1] \tag{19}$$

were considered. But if we put $s_j := 1/x_j$, then

$$\widetilde{v_j}(x) := \left(\frac{1}{x - s_j}\right)^m, \quad |s_j| > 1, \quad x \in [-1, 1] \tag{20}$$

and v_j of (19) span the same spaces. We could even admit $s_1 := \infty$ which would lead to $\widetilde{v_1} := 1$ corresponding to $x_1 = 0$. It is actually not so easy, to show that v_j of (19) generate real Haar spaces on $[-1, 1]$, MEINARDUS & WALZ [13, $\geq$ 2002, section 3.5].

Though the final result (Theorem 20) is at a first glance surprising, there is another result mentioned by HAYMAN [7, 1964, Theorem 2.6, p. 48], saying that two meromorphic functions f_1, f_2 in the complex plane coincide, if $\{z : f_1(z) = a\} = \{z : f_2(z) = a\}$ for five different values of a, where five cannot be replaced by a smaller number.

§4. Appendix: Some details with respect to Lemma 14

Let us recall that G analytic on $\mathbb{C}\setminus\{0\}$ has no zeros (Lemma 11) and that the origin is possibly a pole of G, but not an essential singularity (Lemma 12). In (3) we defined the function F, which played an important role at various places as follows:

$$F(z,t,s) := \frac{G(z-t)}{G(z-s)}, \quad |z| \leq 1, |t| > 1, |s| > 1, s \neq t. \tag{3}$$

Since G may be regarded as a meromorphic function on all of $\mathbb{C}$ this applies to F, too. In addition, we have $F(0,t,s) \neq 0$, $F(0,t,s) \neq \infty$.

Now, let f be any meromorhic function on $\mathbb{C}$ with $f(0) \neq 0, f(0) \neq \infty$. We define the following quantities:

(1.) $\log^+ \alpha := \max(0, \log \alpha)$ for all $\alpha \geq 0$, in particular, $\log^+ 0 = 0$,

(2.) $M(r,f) := \max_{|z|=r} |f(z)|$, $r \geq 0$,

(3.) $n(r,f) := \#\{z : f(z) = 0, |z| < r\}$ multiplicities counted, where $\#$ stands for *number of*,

(4.) $N(r,f) := \int_0^r \frac{n(\sigma,f)}{\sigma}\,\mathrm{d}\sigma$,

(5.) $m(r,f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{\mathrm{i}t})|\,\mathrm{d}t$,

(6.) $T(r,f) := m(r,f) + N(r,\frac{1}{f})$, the so-called Nevanlinna characteristic,

(7.) let h be positive on the non-negative real axis. Then $\varrho(h) := \overline{\lim}_{r\to\infty} \frac{\log^+ h(r)}{\log r}$ is called the order of h. This number may be infinite, e. g. $h(r) := \exp(e^r)$, or finite, e. g. $h(r) = r^p, p \in \mathbb{Z}$ yields $\varrho(h) = \overline{\lim}_{r\to\infty} \frac{\log^+ r^p}{\log r} = \begin{cases} p & \text{for } p > 0, \\ 0 & \text{for } p \leq 0. \end{cases}$
If the order is finite, then one can show that there are constants C, D such that
$\varrho(h) = \inf\{A : h(r) < Ce^{Ar} + D \text{ for all } r > 0\}$.

(8.) $\varrho(f) := \varrho(T(r,f))$ is defined as the order of a meromorhic function f on $\mathbb{C}$.

  (8.1) If f is entire, then $\varrho(f) := \varrho(m(r,f))$, since in this case $n(r,\frac{1}{f}) = 0$, and therefore also $N(r,\frac{1}{f}) = 0$. In addition, one can show that in this case
  $T(r,f) \leq \log M(r,f) \leq \frac{R+r}{R-r}\, T(R,f)$, $R > r$.

  (8.2) As an easy exercise we find $\varrho(z^p) = 0$ for all $p \in \mathbb{N}$.

  (8.3) In the case $f(z) := z^{-p}$, $p \in \mathbb{N}$ we have $m(r,f) = 0$, $N(r,\frac{1}{f}) = N(r,z^p) = p\log r$ and thus, $\varrho(f) = \lim_{r\to\infty} \frac{\log^+ p \log r}{\log r} = 0$.

  (8.4) For the entire function $f(z) := \exp(z^p)$, $p \in \mathbb{N}$ we have $\varrho(f) := \varrho(m(r,f))$ and $m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |\exp(r^p e^{\mathrm{i}pt})|\,\mathrm{d}t = \log^+ e^{r^p \cos pt}$

$= r^p \cos^+ pt$, where $\cos^+ \alpha := \max(0, \cos\alpha)$. Now, $\int_0^{2\pi} \cos^+ pt\,\mathrm{d}t = 2p\sin\frac{\pi}{2p}$, and therefore $\varrho(\exp(z^p)) = \lim_{r\to\infty} \frac{\log^+ r^p \frac{p}{\pi}\sin\frac{\pi}{2p}}{\log r} = p$.

(9.) The first fundamental theorem by R. Nevanlinna says: $T(r,f) = T(r, \frac{1}{f-a}) + O(1)$ for all $a \in \mathbb{C}$.

If we apply the above results to $f(z) := F(z,t,s)$ we obtain:

(I.) $n(r, F(z,t,s) - \mu) \leq M + 16r^2 \Rightarrow N(r, F(z,t,s) - \mu)$ is of order at most two for all μ,

(II.) Cartan: $T(r,f) = \frac{1}{2\pi}\int_0^{2\pi} N(r, f(re^{it}) - e^{it})\,\mathrm{d}t + \log^+ |f(0)| \Rightarrow F(z,t,s)$ is of order at most two,

(III.) the function

$$H(z,t,s) := \frac{(z-s)^m}{(z-t)^m} F(z,t,s) = \frac{(z-s)^m}{(z-t)^m}\frac{G(z-t)}{G(z-s)}$$

is an entire, non vanishing function of order $\varrho \leq 2 \Rightarrow \varrho = 0$ or $\varrho = 1$ or $\varrho = 2 \Rightarrow F(z,t,s)$ is of the form (5).

More details can be found in the beginning two chapters of the book by Hayman [7, 1964].

Acknowledgments. We would like to thank Professor Günter Meinardus, University of Mannheim, who brought the paper by Rahman & Ruscheweyh to our attention and made available parts of a developing book by Meinardus & Walz. We also would like to thank Professor Ron B. Guenther, Oregon State University, Corvallis, Oregon for carefully reading our manuscript.

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