Geometrical insights from supergravity constructions

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Introduction

- The **scalar**s in **field** theories take their values in a differentiable manifold $\mathcal{T}$, called the *target space*.

$$\phi = (\phi^\mu) : \Sigma \to \mathcal{T}$$

- The kinetic energy term of the scalars in the Lagrangian $\mathcal{L}$ of the theory defines a **metric on the target space**.

$$\mathcal{S} = \int_\Sigma \mathcal{L} = \int_\Sigma -g_{\mu\bar{\nu}} \partial_a \phi^\mu \partial^a \bar{\phi}^\nu + \dots .$$

- **Supersymmetry** leads to restrictions on the target space metric depending on the (dim. of the) space-time manifold Σ and on the number of supersymmetry generators, e.g.: $g_{\mu\bar{\nu}} = \frac{\partial^2 K(\phi,\bar{\phi})}{\partial\phi^\mu \partial\bar{\phi}^\nu}$.

- **Dimensional reduction** relates supersymmetric theories of different space-time dimensions and hence their corresponding scalar geometries.

	global $\mathcal{N}=2$ SUSY	local $\mathcal{N}=2$ SUSY
5d vector multiplets	**ASR**	**PSR**
4d vector multiplets	**ASK**	**PSK**
3d (4d) hypermultiplets	**HK**	**QK**

S=special

K=Kähler

R=real

A=affine

P=projective

H=hyper

Q=quaternionic

- **Completeness** in supergravity constructions:
  Local r-map:
  From PSR to PSK manifolds
  Local c-map:
  From PSK to QK manifolds
  r- and c-map preserve completeness!

- **Classification** of complete PSR surfaces
  $\Rightarrow$ 1-parameter family of complete 16-dimensional QK manifolds

- One-loop deformation of c-map and **HK/QK correspondence**
  Rigid c-map:
  From ASK to HK manifolds
  Local c-map:
  From PSK to QK manifolds
  HK/QK correspondence relates rigid and local **c-map+one-loop deformation**

- **Completeness**

[CHM] V. Cortés, X. Han, T. Mohaupt,
Completeness in supergravity constructions,
Comm. Math. Phys. **311** (2012), no. 1, 191–213.

[CDL] V. Cortés, M. D–, D. Lindemann,
Classification of complete projective special real surfaces,
arXiv:1302.4570.

- **HK/QK correspondence**

[ACM] D. V. Alekseevsky, V. Cortés, T. Mohaupt,
Conification of Kähler and hyper-Kähler manifolds,
Comm. Math. Phys. (accepted) (2013).

[ACDM] D. V. Alekseevsky, V. Cortés, M. D–, T. Mohaupt,
Quaternionic Kähler metrics associated with special Kähler manifolds,
arXiv:1305.3549.

Completeness in supergravity constructions

Dimensional reduction of $\mathcal{N} = 2$ supergravity $5d \to 4d \to 3d$
$\Rightarrow$

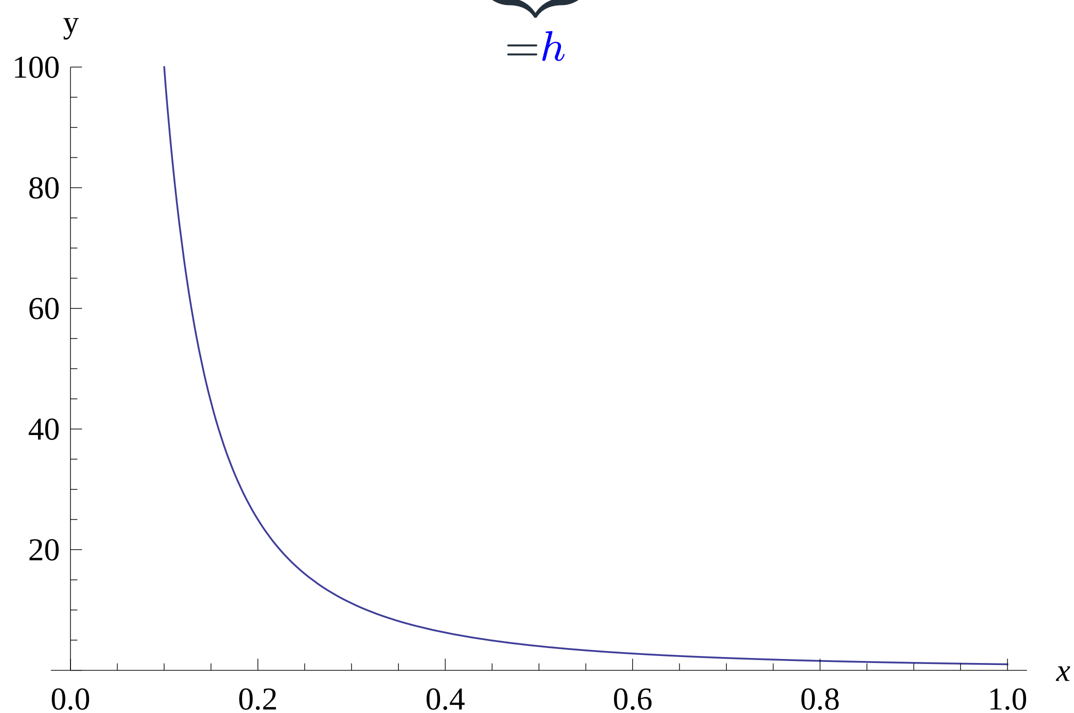
$$\text{PSR} \xrightarrow{\text{r-map}} \text{PSK} \xrightarrow{\text{c-map}} \text{QK}$$

Theorem 1 [CHM]
Let (M_1, g_1) be a complete Riemannian manifold and $(g_2(p))_p$ a smooth family of G-invariant Riemannian metrics on a homogeneous manifold $M_2 = G/K$, depending on a parameter $p \in M_1$. Then the Riemannian metric $g = g_1 + g_2$ on $M = M_1 \times M_2$ is complete. Moreover, the action of G on M_2 induces an isometric action of G on (M, g).

[CHM] Cortés, Han, Mohaupt,
Completeness in supergravity constructions,
Comm. Math. Phys. **311** (2012), no. 1, 191–213.

Definition 1 Let h be a homogeneous cubic polynomial in n variables with real coefficients and let $U \subset \mathbb{R}^n \setminus \{0\}$ be an $\mathbb{R}^{>0}$-invariant domain such that $h|_U > 0$ and such that $g_{\mathcal{H}} := -\partial^2 h|_{\mathcal{H}}$ is a Riemannian metric on the hypersurface $\mathcal{H} := \{h = 1\} \subset U$. Then $(\mathcal{H}, g_{\mathcal{H}})$ is called a **projective special real** (PSR) manifold.

Example 1 $\mathcal{H} = \{(x,y) \in \mathbb{R}^2 | \underbrace{x^2 y}_{=h} = 1, x > 0\}, U = (\mathbb{R}^{>0})^2 \subset \mathbb{R}^2$.



Definition 2 A **conical affine special Kähler manifold** (M, J, g_M, ∇, ξ) is a pseudo-Kähler manifold (M, J, g_M) endowed with a flat torsionfree connection ∇ and a vector field ξ such that

i) $\nabla\omega = 0$, where $\omega := g_M(J\cdot, \cdot)$ is the Kähler form,
ii) $(\nabla_X J)Y = (\nabla_Y J)X$ for all $X, Y \in \Gamma(TM)$,
iii) $\nabla\xi = D\xi = \mathrm{Id}$, where D is the Levi-Civita connection,
iv) g_M is positive definite on $\mathcal{D} = \mathrm{span}\{\xi, J\xi\}$ and negative definite on $\mathcal{D}^\perp$.

Locally, there exist so-called *conical special holomorphic coordinates* z^I, $I = 0, \ldots, n$ and a holomorphic function $F(z)$, homogeneous of degree 2 such that

$$g_M = N_{IJ} dz^I d\bar{z}^J, \quad \xi = z^I \partial_I + \bar{z}^I \partial_{\bar{I}},$$

where $N_{IJ} := 2\mathrm{Im}\, F_{IJ}(z)$. The Kähler potential is
$K := r^2 := g_M(\xi, \xi) = z^I N_{IJ} \bar{z}^J = i(z^I \bar{F}_I - \bar{z}^I F_I)$.

Let (M, J, g_M, ∇, ξ) be a CASK manifold. Then $J\xi$ is a holomorphic Killing field and the Kähler reduction of (M, J, g_M) w.r.t. $J\xi$ with the choice of level set $S := \{g_M(\xi, \xi) = 1\}$ gives a Kähler manifold $\{\bar{M} = S/S^1_{J\xi}, -g_{\bar{M}}, \bar{J}\}$.

$$M_{\text{CASK}} \;\big\downarrow \mathbb{R}^{>0}\; S_{\text{Sasaki}} \;\big\downarrow S^1\; \bar{M}_{\text{PSK}}$$

Definition 3 $(M, g_{\bar{M}}, \bar{J})$ is called a **projective special Kähler manifold**.

If z^I, $I = 0, \ldots, n$ are conical special holomorphic coordinates on M, then $X^\mu := \frac{z^\mu}{z^0}$, $\mu = 1, \ldots, n$ define a local holomorphic coordinate system on $\bar{M}$. The Kähler potential for $g_{\bar{M}}$ is $\mathcal{K} := -\log X^I N_{IJ}(X)\bar{X}^J$, where $X := (X^0, \ldots, X^n)$ with $X^0 := 1$.

Let $(\mathcal{H}, -\partial^2 h|_{\mathcal{H}})$ be a projective special real manifold.
Then $U = \mathbb{R}^{>0} \cdot \mathcal{H}$ and $-\partial^2 h$ is a Lorentzian metric on U.

We define $M := \mathbb{R}^n + iU \subset \mathbb{C}^n$ with holomorphic coordinates $(X^\mu) = (y^\mu + ix^\mu) \in \mathbb{R}^n + iU$ and endow it with a Kähler metric $g_M = \frac{\partial^2 K}{\partial X^\mu \partial \bar{X}^\nu} dX^\mu d\bar{X}^\nu$ defined by the Kähler potential

$$\mathcal{K}(X, \bar{X}) := -\log h(x) = -\log h(\operatorname{Im} X).$$

Definition 4 The correspondence $(\mathcal{H}, g_{\mathcal{H}}) \mapsto (M, g_M)$ is called the **local r-map**.

$$(M, g_M) \approx (\mathcal{H} \times \overbrace{\mathbb{R} \times \mathbb{R}^n}^{G}, g_{\mathcal{H}} + \overbrace{dr^2 - \frac{\partial^2 \log h(x)}{\partial x^\mu \partial x^\nu} dy^\mu dy^\nu}^{g_G}).$$

In terms of the prepotentials h and F, the local r-map is given by

$$h(x^\mu) \mapsto F(z^I) = \frac{h(z^\mu)}{z^0}.$$

$$g_G = \frac{1}{4\rho^2}d\rho^2 + \frac{1}{4\rho^2}(d\tilde{\phi} + \sum(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I))^2 + \frac{1}{2\rho}\sum \mathcal{I}_{IJ}(p) d\zeta^I d\zeta^J$$
$$+\frac{1}{2\rho}\sum \mathcal{I}^{IJ}(p)(d\tilde{\zeta}_I + \mathcal{R}_{IK}(p)d\zeta^K)(d\tilde{\zeta}_J + \mathcal{R}_{JL}(p)d\zeta^L),$$

where $(\rho, \tilde{\phi}, \tilde{\zeta}_I, \zeta^I) \in \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$.

$$\mathcal{N}_{IJ} := \mathcal{R}_{IJ} + i\mathcal{I}_{IJ} := \bar{F}_{IJ} + i\frac{\sum_K N_{IK}z^K \sum_L N_{JL}z^L}{\sum_{IJ} N_{IJ}z^I z^J}, \; N_{IJ} := 2\mathrm{Im}F_{IJ}.$$

Definition 5 Let $(\bar{M}, g_{\bar{M}})$ be a $2n$-dim. projective special Kähler domain. The correspondence

$$(\bar{M}, g_{\bar{M}}) \mapsto (\bar{N} = \bar{M} \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}, g_{\bar{N}} = g_{\bar{M}} + g_G)$$

is called the **local c-map**.

(N, g_N) is quaternionic Kähler [FS].

[FS] Ferrara, Sabharwal, Nucl. Phys. **B332** (1990), 317–332.

Dimensional reduction of $\mathcal{N} = 2$ supergravity $5d \to 4d \to 3d$
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Theorem 2 [CHM]
Let (M_1, g_1) be a complete Riemannian manifold and $(g_2(p))_p$ a smooth family of G-invariant Riemannian metrics on a homogeneous manifold $M_2 = G/K$, depending on a parameter $p \in M_1$. Then the Riemannian metric $g = g_1 + g_2$ on $M = M_1 \times M_2$ is complete. Moreover, the action of G on M_2 induces an isometric action of G on (M, g).

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$$\text{complete PSR} \xrightarrow{\text{r-map}} \text{complete PSK} \xrightarrow{\text{c-map}} \text{complete QK}$$

Corollary 1 [Cortés, Han, Mohaupt 2012]
Combining the supergravity r- and c-map, one obtains a complete $4m + 8$-dimensional quaternionic Kähler manifold from each complete m-dimensional projective special real manifold.

Theorem 3 [CDL ’13]

There exist precisely five discrete examples and a one-parameter family of complete projective special real surfaces, up to isomorphism:

a) $\{(x,y,z) \in \mathbb{R}^3 \,|\, xyz = 1,\ x > 0,\ y > 0\}$,
b) $\{(x,y,z) \in \mathbb{R}^3 \,|\, x(xy - z^2) = 1,\ x > 0\}$,
c) $\{(x,y,z) \in \mathbb{R}^3 \,|\, x(yz + x^2) = 1,\ x < 0,\ y > 0\}$,
d) $\{(x,y,z) \in \mathbb{R}^3 \,|\, z(x^2 + y^2 - z^2) = 1,\ z < 0\}$,
e) $\{(x,y,z) \in \mathbb{R}^3 \,|\, x(y^2 - z^2) + y^3 = 1,\ y < 0,\ x > 0\}$,
f) $\{(x,y,z) \in \mathbb{R}^3 \,|\, y^2 z - 4x^3 + 3xz^2 + bz^3 = 1,\ z < 0,\ 2x > z\}$, where $b \in (-1,1) \subset \mathbb{R}$.

[CDL] Cortés, D–, Lindemann, *Classification of complete projective special real surfaces*, arXiv:1302.4570.

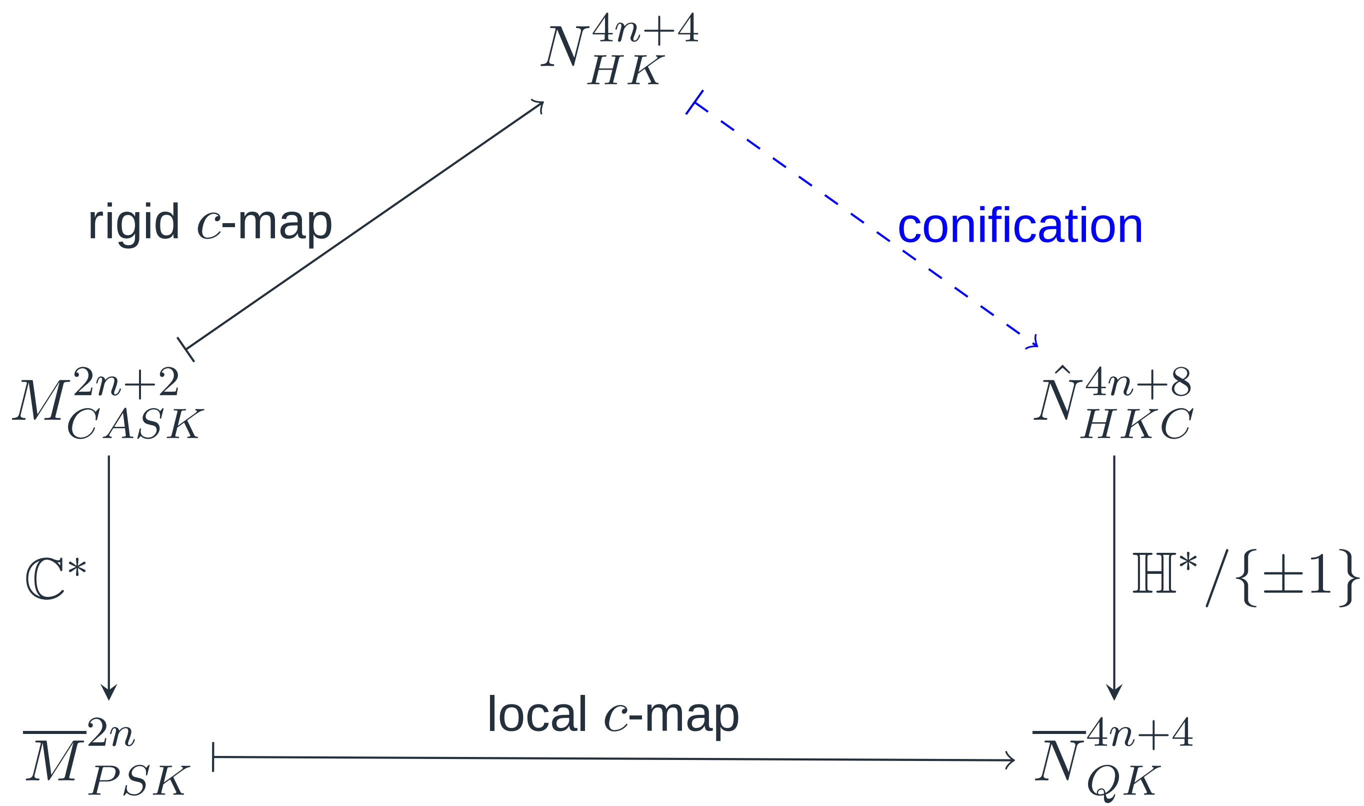
Classification of complete PSR surfaces

+ local r-map

+ local c-map

$\Rightarrow$ **Explicit 1-parameter family of complete 16-dim. quaternionic Kähler metrics!**

One-loop deformation of the c-map and HK/QK-correspondence


N^{4n+4}_{HK}
rigid c-map
conification
M^{2n+2}_{CASK}
$\hat{N}^{4n+8}_{HKC}$
$\mathbb{C}^*$
$\mathbb{H}^*/\{\pm 1\}$
local c-map
$\overline{M}^{2n}_{PSK}$
$\overline{N}^{4n+4}_{QK}$

Let $(M \subset \mathbb{C}^{n+1}, J, g_M, \xi)$ be a CASK manifold, globally described by the holomorphic prepotential $F(z)$.
The real coordinates $(q^a) := (x^I, y_J) := (\operatorname{Re} z^I, \operatorname{Re} F_J(z))$ on M are ∇-affine and fulfill $\omega_M = -2dx^I \wedge dy_I$.
Let $(p_a) := (\tilde{\zeta}_I, \zeta^J)$ be real functions on $N := T^*M$ such that (q^a, p_b) is a system of canonical coordinates.

Proposition 1 In the above coordinates $(z^I, \tilde{\zeta}_J, \zeta^K)$, the hyper-Kähler structure on $N = T^*M$ obtained from the rigid c-map is given by

$$
\begin{aligned}
g_N &= \sum dz^I N_{IJ} d\bar{z}^J + \sum A_I N^{IJ} \bar{A}_J, \\
\omega_1 &= \frac{i}{2} \sum N_{IJ} dz^I \wedge d\bar{z}^J + \frac{i}{2} \sum N^{IJ} A_I \wedge \bar{A}_J, \\
\omega_2 &= -\frac{i}{2} \sum (d\bar{z}^I \wedge \bar{A}_I - dz^I \wedge A_I), \\
\omega_3 &= \frac{1}{2} \sum (dz^I \wedge A_I + d\bar{z}^I \wedge \bar{A}_I),
\end{aligned}
$$

where $A_I := d\tilde{\zeta}_I + \sum_J F_{IJ}(z) d\zeta^J$ $(I = 0, \ldots, n)$ are complex-valued one-forms on N and $\omega_\alpha = g_N(J_\alpha \cdot, \cdot)$.

Let (N, g, J_1, J_2, J_3) be a $4m$-dim. (pseudo-)hyper-Kähler manifold with a J_1-holomorphic Killing vector field that fulfills $\mathcal{L}_Z J_2 = -2J_3$. Then one can (locally) construct a conical $4m+4$-dim. hyper-Kähler manifold $\hat{N}$ such that N is obtained from $\hat{N}$ via a hyper-Kähler quotient [ACM] (see also [Ha]). This constructions depends on a real parameter c.

Using the Swann-bundle construction of a hyper-Kähler cone as a $\mathbb{H}/\{\pm 1\}$-bundle over a quaternionic Kähler manifold, we get a correspondence between a HK manifold and a 1-parameter family of QK manifolds which is called HK/QK correspondence.

[ACM] Alekseevsky, Cortés, Mohaupt, *Conification of Kähler and hyper-Kähler manifolds*, Comm. Math. Phys. (accepted) (2013).

[Ha] Haydys, J. Geom. Phys. **58** (2008), no.3, 293–306.

HK/QK correspondence

Let (M, g, J_1, J_2, J_3) be a pseudo-hyper-Kähler manifold, Z a time-like or space-like J_1-holomorphic Killing vector field with $\mathcal{L}_Z J_2 = -2J_3$ and $f \in C^\infty(M)$ such that $df = -\omega(Z, \cdot)$ and such that f and $f_1 = f - \frac{g(Z,Z)}{2}$ are non-vanishing.

Let $\pi : P \to M$ be an S^1-principal bundle with a connection η having curvature $d\eta = \omega - \frac{1}{2}d\beta$, where $\beta := g(Z, \cdot)$.

We endow P with the pseudo-Riemannian metric, vector field and one-forms

$$g_P := \frac{2}{f_1}\eta^2 + g \qquad Z_1^P := Z + (f_1 - \eta(Z))X_P,$$

$$\theta_0^P := -\frac{1}{2}df \qquad \theta_1^P := \eta + \frac{1}{2}\beta,$$

$$\theta_2^P := \frac{1}{2}\omega_3(Z, \cdot) \qquad \theta_3^P := -\frac{1}{2}\omega_2(Z, \cdot).$$

Theorem 4 The tensor field

$$\tilde{g}_P := g_P - \frac{2}{f}\sum_{a=0}^{3}(\theta_a^P)^2$$

on P is invariant under Z_1^P and has one-dimensional kernel $\mathbb{R}Z_1^P$. Let M' be a submanifold of P which is transversal to the vector field Z_1^P. Then

$$g' := \frac{1}{2|f|}\tilde{g}_P|_{M'}$$

is a (pseudo-)quaternionic Kähler metric on M'.

There exists a canonical vector field Z on any HK manifold (N, g_N, J_1, J_2, J_3) obtained from the rigid c-map that fulfills the assumptions of the HK/QK correspondence [ACM].

Theorem 5 [ACDM '13]
The Ferrara-Sabharwal metric and its one-loop deformation can be obtained from (N, g_N, J_1, J_2, J_3) by applying the HK/QK-correspondence with respect to Z.

This gives a new (mathematical) proof that the Ferrara-Sabharwal metric and its one-loop deformation are indeed quaternionic Kähler.

[ACDM] Alekseevsky, Cortés, D–, Mohaupt, *Quaternionic Kähler metrics associated with special Kähler manifolds*, arXiv:1305.3549.

Deformed Ferrara-Sabharwal metric:

$$\begin{aligned} g^c_{FS} = \frac{\rho + c}{\rho} g_{\bar{M}} &+ \frac{1}{4\rho^2}\frac{\rho+2c}{\rho+c} d\rho^2 + \frac{1}{2\rho} dp_a \hat{H}^{ab} dp_b \\ &+ \frac{1}{4\rho^2}\frac{\rho+c}{\rho+2c}\left(d\tilde{\phi} + \sum (\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I) + c d^c \mathcal{K}\right)^2 \\ &+ \frac{2c}{\rho^2} e^{\mathcal{K}} \left| \sum (X^I d\tilde{\zeta}_I + F_I(X) d\zeta^I) \right|^2, \end{aligned}$$

where $(X^I, \rho, \tilde{\phi}, \tilde{\zeta}_J, \zeta^K) \in \bar{M} \times \mathbb{R}^* \times \mathbb{R}^{4n+3}$ and

* g^c_{FS} is positive definite for $\rho > \max\{0, -2c\}$,
* g^c_{FS} is of signature $(4n, 4)$ for $-c < \rho < -2c$, $c < 0$,
* g^c_{FS} is of signature $(4, 4n)$ for $-c < \rho < 0$, $c > 0$.

K/K correspondence

Let (M, g, J) be a pseudo-Kähler manifold, Z a time-like or space-like holomorphic Killing vector field and $f \in C^\infty(M)$ such that $df = -\omega(Z, \cdot)$ and such that f and $f_1 = f - \frac{g(Z,Z)}{2}$ are non-vanishing.
Let $\pi : P \to M$ be an S^1-principal bundle with a connection η having curvature $d\eta = \omega - \frac{1}{2} d\beta$, where $\beta := g(Z, \cdot)$.
We endow P with the pseudo-Riemannian metric, vector field and one-forms

$$
\begin{aligned}
g_P &:= \frac{2}{f_1}\eta^2 + g \\
Z^P &:= Z + (f_1 - \eta(Z))X_P \\
\theta_0^P &:= -\frac{1}{2} df \\
\theta_1^P &:= \eta + \frac{1}{2}\beta.
\end{aligned}
$$

Theorem 6 The tensor field

$$\tilde{g}_P := g_P - \frac{2}{f} \sum_{a=0}^{1} (\theta_a^P)^2$$

on P is invariant under Z^P and has one-dimensional kernel $\mathbb{R}Z^P$. Let M' be a submanifold of P which is transversal to the vector field Z^P. Then

$$g' := \frac{1}{2|f|} \tilde{g}_P|_{M'}$$

is a (pseudo-)Kähler metric on M'.

Here, we consider $M = \{z = re^{i\phi}\} = \mathbb{C}^*$ with its standard complex structure and metric $g = dz\, d\bar{z} = dr^2 + r^2 d\phi^2$, endowed with the holomorphic Killing field $Z := 2(iz\partial_z - i\bar{z}\partial_{\bar{z}}) = 2\partial_\phi$. The Kähler form is given by $\omega = \frac{i}{2} dz \wedge d\bar{z} = r dr \wedge d\phi$.

We have $\omega(Z, \cdot) = -2r dr \overset{!}{=} -df$ and choose $f = r^2 > 0$. Then $f_1 = f - \frac{g(Z,Z)}{2} = -r^2 < 0$. We consider

$$P = M \times S^1, \quad S^1 = \{e^{is} | s \in \mathbb{R}\}$$

with the connection $\left[\beta = g(Z, \cdot) = 2r^2 d\phi\right]$

$$\eta = ds - \frac{1}{2} r^2 d\phi, \quad d\eta = -\omega = \omega - \frac{1}{2} d\beta.$$

We have $Z^P = 2\partial_\phi$ and choose the transversal manifold $M' = \{\phi = 0\} \subset \mathbb{C}^*$, i.e. $M' = \{r | r \in \mathbb{R}^{>0}\}$.
We have $\beta|_{M'} = 0$, $\theta_0^P = -\frac{1}{2}df = -rdr$, $\theta_1^P|_{M'} = ds$, i.e.

$$(\theta_0^P)^2 + (\theta_1^P)^2|_{M'} = r^2dr^2 + ds^2.$$

$$g_P|_{M'} = \frac{2}{f_1}\eta^2|_{M'} + g|_{M'} = -\frac{2}{r^2}ds^2 + dr^2.$$

$$\tilde{g}_P|_{M'} = g_P|_{M'} - \frac{2}{f}\sum_{a=0}^{1}(\theta_a^P)^2|_{M'} = -\frac{4}{r^2}ds^2 - dr^2.$$

$$\Rightarrow -g' = -\frac{1}{|f|}\tilde{g}_P|_{M'} = \frac{1}{4\rho^2}(d\tilde{\phi}^2 + d\rho^2) = g_{\mathbb{C}H^1},$$

where $\tilde{\phi} := -4s$, $\rho := r^2$.

Thank you!