Exercise Sheet 12, Advanced Algebra, Summer Semester 2017. Hints to exercise sheet 12

(Ehud Meir and Christoph Schweigert)

---

1. (a) We have seen that a projective resolution for $\mathbb{Z}/m$ is given by

$$0 \to \mathbb{Z} \stackrel{m}{\to} \mathbb{Z} \to \mathbb{Z}/m \to 0.$$

A similar resolution exists for $\mathbb{Z}/n$. To find the lifting we consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \stackrel{m}{\longrightarrow} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/m & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \stackrel{n}{\longrightarrow} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/n & \longrightarrow & 0 \end{array}$$

The maps ϕ_0 and ϕ_1 are given by multiplication by some numbers. A direct calculation, by chasing the image of $1 \in \mathbb{Z}$ shows us now that $\phi_0(x) = rx$ and $\phi_1(x) = ax$ where $a = \frac{rm}{n}$.

(b) To calculate the action on the derived functors, we apply the functors to the resolutions (without $\mathbb{Z}/m$ and $\mathbb{Z}/n$) and calculate homology. We begin with $\otimes\mathbb{Z}/k$ and its derived functors Tor_n^R (here $R = \mathbb{Z}$). We get the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}/k & \stackrel{m}{\longrightarrow} & \mathbb{Z}/k & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle r} & & \\ 0 & \longrightarrow & \mathbb{Z}/k & \stackrel{n}{\longrightarrow} & \mathbb{Z}/k & \longrightarrow & 0 \end{array}$$

The zeroth homology of the first line is just

$$Tor_0^R(\mathbb{Z}/m, \mathbb{Z}/k) = \mathbb{Z}/m \otimes \mathbb{Z}/k = \mathbb{Z}/gcd(m,k)$$

and of the second line is

$$Tor_0^R(\mathbb{Z}/n, \mathbb{Z}/k) = \mathbb{Z}/n \otimes \mathbb{Z}/k = \mathbb{Z}/gcd(n,k).$$

The induced map is just multiplication by r.
The first homology of the first line is

$$Tor_1^R(\mathbb{Z}/m, \mathbb{Z}/k) = Ker(\mathbb{Z}/k \stackrel{m}{\to} \mathbb{Z}/k) \cong \langle k/gcd(k,m) \rangle \cong \mathbb{Z}/gcd(k,m).$$

Similarly the first homology of the second line is

$$Tor_1^R(\mathbb{Z}/n, \mathbb{Z}/k) = Ker(\mathbb{Z}/k \xrightarrow{n} \mathbb{Z}/k) \cong \langle k/gcd(k,n) \rangle \cong \mathbb{Z}/gcd(k,n).$$

Let us write $k/gcd(k,m) = k'$ and $k/gcd(k,n) = k''$. It follows that the map induced by ϕ will send the generator k' of the first homology group of the first line to the element ak' inside the first homology group of the second line.

For the ext groups, we apply the contravariant functor $Hom_R(-, \mathbb{Z}/k)$ to the diagram. We get now the diagram:

$$\begin{array}{ccccccc}
0 & \longleftarrow & \mathbb{Z}/k & \xleftarrow[m]{} & \mathbb{Z}/k & \longleftarrow & 0 \\
 & & \uparrow{\scriptstyle a} & & \uparrow{\scriptstyle r} & & \\
0 & \longleftarrow & \mathbb{Z}/k & \xleftarrow[n]{} & \mathbb{Z}/k & \longleftarrow & 0
\end{array}$$

The zeroth homology in the first line is now

$$Ext_R^0(\mathbb{Z}/m, \mathbb{Z}/k) = Ker(\mathbb{Z}/k \xrightarrow{m} \mathbb{Z}/k) \cong \mathbb{Z}/gcd(k,m)$$

and the zeroth homology of the second line is calculated similarly. The induced map is then induced by multiplication by r. The first homology group of the first line is then

$$Ext_R^1(\mathbb{Z}/m, \mathbb{Z}/k) = \mathbb{Z}/k/m\mathbb{Z}/k \cong \mathbb{Z}/gcd(m,k).$$

The induced map $\mathbb{Z}/gcd(m,k) \to \mathbb{Z}/gcd(n,k)$ is induced by multiplication by a.

2. For an object $X \in \mathcal{C}$ we write C_X^n for the complex

$$\cdots \to X \xrightarrow{Id} X \to 0 \to \cdots$$

where the copies of X are in degrees n and $n-1$. We write D_X^n for the complex

$$\cdots \to 0 \to X \to 0 \to \cdots$$

where X is concentrated in degree n.

(a) We first prove that C_X^n is projective if and only if X is projective. We have calculated in class and saw that we have an isomorphism of functors $Hom_{Ch(\mathcal{C})}(C_X^n, C) \cong Hom_{\mathcal{C}}(X, C_n)$. If X is projective then $Hom_{\mathcal{C}}(X, -)$ is exact and as a result $Hom_{Ch(\mathcal{C})}(C_X^n, -)$ is exact, which means that C^n_X is projective. If X is not projective we use the same argument, using the fact that if

$$\begin{array}{ccccc} & & X & & \\ & & \downarrow & & \\ Y & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

is a diagram in $\mathcal{C}$ which cannot be completed, then we get a diagram

$$\begin{array}{ccccc} & & C^n_X & & \\ & & \downarrow & & \\ C^n_Y & \longrightarrow & C^n_Z & \longrightarrow & 0 \end{array}$$

in $Ch(\mathcal{C})$ which cannot be completed.

(b) We first prove that $d_1 : X_1 \to X_0$ is surjective. We write $M = X_0/d_1(X_1)$. we have a canonical surjection $C^0_M \to D^0_M$. We have a chain map from C to D^0_M given by the natural projection in degree 0, and this map is liftable to a map from C to C^0_M. As we have seen in class, this is possible only if $M = 0$, which means that d_1 is surjective. We next prove that the map d_1 splits. We define a chain map $C \to C^1_{X_0}$ which in degree 1 is given by $d_1 : X_1 \to X_0$ and in degree 0 is given by $Id : X_0 \to X_0$. We use the canonical surjection $C^1_{X_1} \to C^1_{X_0}$ and use the fact that C is projective. This gives us a splitting of d_1. This enables us to write $X_1 \cong Ker(d_1) \oplus X_0$. Since the image of d_2 is contained in $Ker(d_1)$, we get a direct sum decomposition of C as

$$C = (\cdots \to 0 \to X_n \to \cdots \to X_2 \to Ker(d_1) \to 0 \to \cdots)\oplus$$

$$(\cdots 0 \to X_0 \to X_0 \to \cdots).$$

Since C is projective the two direct summands are projective as well, and we continue by induction.