Exercise Sheet 10, Advanced Algebra, Summer Semester 2017- Hints and partial solutions

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1. (c): The fields L_i are quotients of the ring $\mathbb{Q}C_n$. They are therefore generated over $\mathbb{Q}$ by an element whose order divides n. Such fields will therefore necessarily be of the form $\mathbb{Q}(\zeta_r)$ where $r|n$ and ζ_r is a primitive r-th root of unity.

2. We have calculated and seen that Q_8 has 5 conjugacy classes:

$$\{1\},\{-1\},\{\pm i\},\{\pm j\},\{\pm k\}.$$

There are therefore 5 irreducible representations. Using the equation $\sum_i d_i^2 = |G| = 8$, we saw that the dimensions must be $1,1,1,1,2$. For the one dimensional representations, we have calculated all homomorphisms $G \to GL_1(\mathbb{C})$. We then filled up the last column by using the orthogonality relations. We got the following character table:

conj.	V_1	V_2	V_3	V_4	V_5
$\{1\}$	1	1	1	1	2
$\{-1\}$	1	1	1	1	-2
$\{\pm i\}$	1	1	-1	-1	0
$\{\pm j\}$	1	-1	1	-1	0
$\{\pm k\}$	1	-1	-1	1	0

3. For the dihedral group, $D_8 = \langle x,y | x^2 = y^4 = 1, xyx^{-1} = y^{-1}\rangle$, the conjugacy classes are $\{1\},\{y^2\},\{y,y^3\},\{x,xy^2\},\{xy,xy^3\}$. Again, the dimensions of the irreducible representations are $1,1,1,1,2$. A similar analysis, using the fact that $y^2 \in [G,G]$ reveals the fact that the character table is exactly the same as that of Q_8, even though the two groups are non-isomorphic.

4. (a) We write $X = \{x_1,\dots x_n\}$. We have seen that the matrix which corresponds to $g \in G$ is given by

$$g_{ij} = \begin{cases} 1 & \text{if } g(x_j) = x_i, \\ 0 & \text{else} \end{cases}$$

This implies that $\chi(g) = \sum_{i|g(x_i)=x_i} 1$ which is exactly the number of fixed points of g.

(b) There are (at least) two solutions. One option is to use the equation $dimHom_G(U,V) = (\chi_U, \chi_V)$, and use the fact that $Hom_G(\mathbb{C}, V) = V^G$ has a basis by the elements $\sum_{y \in G \cdot x} e_y$ for the different orbits of G in X. Another option is to write

$$(\chi_V, 1) = \frac{1}{|G|} \sum_{g \in G} \chi(g) = \frac{1}{|G|} \sum_{g \in, x \in X, g(x) = x} 1 =$$

$$\frac{1}{|G|} \sum_{x \in X} |Stab_G(x)| = \frac{1}{|G|} \sum_{[x] \in X/G} |Stab_G(x)||G \cdot x| =$$

$$\frac{1}{|G|} \sum_{[x] \in X/G} |G| = |X/G|.$$

5. (a) We calculate

$$T^2 v = \frac{1}{|G|^2} \sum_{g,h \in G} ghv = \frac{1}{|G|^2} \sum_{g,h \in G} gv = \frac{1}{G} \sum_{g \in G} gv = Tv.$$

For the second equality we have used a change of summation of the form $g \mapsto gh, h \mapsto h$. If $h \in G$ then it holds that $h \cdot Tv = \frac{1}{|G|} \sum_{g \in G} hgv = \frac{1}{|G|} \sum_{g \in G} gv = Tv$. Finally, if $v \in V^G$ then $Tv = \frac{1}{|G|} \sum_{g \in G} gv = \frac{1}{|G|} \sum_{g \in G} v = v$, and so the image is exactly V^G. It is true for every projection that $tr(T) = dim(Im(T))$ (diagonalize!) and so in this case we get $\frac{1}{|G|} \sum_{g \in G} \chi(g) = V^G$.

(b) Let $\{u_1, \ldots u_n\}$ and $\{v_1, \ldots v_m\}$ be bases for U and for V respectively. Then a basis for the space $Hom(U,V)$ is given by the "matrix units" e_{ij} where $e_{ij}(u_k) = \delta_{jk} v_i$. If $g \in G$ acts on V by a matrix A_g and on U by a matrix B_g, then we will get that

$$g \cdot e_{ij} = A_g e_{ij} B_{g^{-1}} = \sum_{k,l} (a_g)_{ki} (b_{g^{-1}})_{jl} e_{kl}.$$

So by using the basis e_{ij} we see that the character value on g will be $\sum_{i,j} (a_g)_{ii} (b_{g^{-1}})_{jj} = \psi(g^{-1}) \chi(g)$.

(c) It holds that $Hom(U,V)^G = Hom_G(U,V)$. By the first part of this exercise and by the formula for the character, the dimension of this space is exactly $\frac{1}{|G|} \sum_{g \in G} \psi(g^{-1}) \chi(g) = (\chi, \psi)$. In particular, we see that the characters of the irreducible representations are orthogonal to one another, and have norm 1.