Exercise Sheet 7, Advanced Algebra, remarks on left over exercises
(Ehud Meir and Christoph Schweigert)

3.

(a) No, for example $\mathbb{Z}/6\mathbb{Z}$, which is not an integral domain.

(b) No, for example $K[X,Y]$ is a subring of $K(X)[Y]$, and $\mathbb{Z}[X]$ is a subring of $\mathbb{Q}[X]$ (why are the subrings not PIDs?)

4. If $x_1 \leq x_2$ we denote by f_{x_1,x_2} the unique morphism in $\mathcal{C}_X$ from x_1 to x_2.

(a) If $F : \mathcal{C}_X \to \mathcal{C}_Y$ is a functor, then by considering the way F acts on objects of the category $\mathcal{C}_X$ we get a function from X to Y. If $x_1 \leq x_2$ in X, then $F(f_{x_1,x_2})$ is a morphism in $Hom_{\mathcal{C}_Y}(f(x_1), f(x_2))$. In particular, this set is non-empty, and we get that $f(x_1) \leq f(x_2)$, which means that f is monotonous. In the other direction, if $f : X \to Y$ is monotonous we define a functor F by $F(x) = f(x)$ for objects in X. Since all morphism sets contain zero or one element the action of F on morphisms is uniquely defined (the action on morphisms is then well defined since f is monotonous).

(b) If $G : \mathcal{C}_{\mathbb{R}} \to \mathcal{C}_{\mathbb{Q}}$ is a left adjoint functor of F, then it corresponds to a function $g : \mathbb{R} \to \mathbb{Q}$ which satisfies that $g(y) \leq x$ if and only if $y \leq f(x)$ for every $x \in \mathbb{Q}$ and $y \in \mathbb{R}$ (to see this, just consider the isomorphism $Hom_{\mathcal{C}_X}(G(y), x) \cong Hom_{\mathcal{C}_Y}(y, F(x))$. From basic calculus we know that such a function g cannot exist: there is, for example, no rational number t such that $\{x \in \mathbb{Q} | t \leq x\} = \{x \in \mathbb{Q} | \sqrt{2} \leq x\}$. For similar reasons, F does not admit a right adjoint functor.