Exercise class, 6.4.17, Advanced Algebra, Summer Semester 2017- some remarks

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1. The following question rose in class today: What examples of rings R do we have so that R is not isomorphic to R^{opp}? I gave one example without proof:

$$R = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a \in \mathbb{Z}, b, c \in \mathbb{Q}\} \subseteq M_2(\mathbb{Q}).$$

I will write a proof here for another ring:

$$R = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a \in \mathbb{Z}[\frac{1}{2}], b \in \mathbb{Z}[\frac{1}{6}], c \in \mathbb{Z}[\frac{1}{3}]\}$$

(Exercise: prove that this is indeed a ring. The ring $\mathbb{Z}[\frac{1}{n}]$ stands for the subring of $\mathbb{Q}$ of all elements of the form $\frac{a}{n^t}$ (where n is a natural number)). Assume that $\phi : R \to R^{opp}$ is an isomorphism of rings. Consider the element $X = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}$. This element satisfies the equation $2X^2 - X = 0$. It is then easy to see that $\phi(X)$ satisfies the same equation. But the only elements in R satisfying this equation are elements of the form $\begin{pmatrix} \frac{1}{2} & b \\ 0 & 0 \end{pmatrix}$ (use linear algebra to prove that!). It follows that $\phi(X) = \begin{pmatrix} \frac{1}{2} & b \\ 0 & 0 \end{pmatrix}$ for some $b \in \mathbb{Z}[\frac{1}{6}]$. The element $Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ satisfies the equation $Y^2 = 0$. Just as with X, it follows that $\phi(Y)$ satisfies the same equation. Since the only elements which satisfy this equation are of the form $\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}$ we can assume that $\phi(Y) = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}$ for some $c \in \mathbb{Z}[\frac{1}{6}]$. It holds that $XY = \frac{1}{2}Y \neq 0$ but $\phi(XY) = \phi(X) \cdot_{opp} \phi(Y) = \phi(Y)\phi(X) = 0$, contradicting the fact that ϕ is an isomorphism. The ring R is therefore not isomorphic to R^{opp}.

2. We proved at the end of the class that if R is an integral domain and $\Phi : R[X] \to R[X]$ is an isomorphism such that $\Phi(r) = r$ for every $r \in R$ then $\Phi(X) = aX + b$ for some $a \in R^\times$ and $b \in R$. The assumption that R is an integral domain is really necessary here: consider for example the case where $R = \mathbb{Z}/4$. Define $\Phi : R[X] \to R[X]$ by $\Phi(X) = X + 2X^2$. Then $\Phi^2(X) = \Phi(X + 2X^2) = X + 2X^2 + 2(X + 2X^2)^2 = X$.