Exercise Sheet 11, Advanced Algebra, Summer Semester 2017. To be discussed on Thursday 6.7.17
(and pending G20 and the progression of the lecture also on 13.7.17)

(Ehud Meir and Christoph Schweigert)

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1. Let $\mathcal{C}$ be an abelian category and $Ch_{\mathcal{C}}$ the associated category of chain complexes. Show that $Ch_{\mathcal{C}}$ is an abelian category.

2. Let $C_\bullet$ a complex of abelian groups such that all groups C_n are *free* abelian groups. Which of the following groups are free? The cycles, the boundary, the homology groups?

3. Find two different projective resolutions of the $\mathbb{Z}$-module $\mathbb{Z}_n$ and find a homotopy equivalence.

4. Let $C_\bullet$ and $D_\bullet$ be chain complexes in an abelian category $\mathcal{C}$ and let $\phi_\bullet : C_\bullet \to D_\bullet$ be a morphism of chain complexes.

   (a) Show that

   $$E_n := C_{n-1} \oplus D_n \qquad d(a,b) = (-da, \phi(a) + db) \quad \text{mit } a \in C_{n-1} \text{ und } b \in D_n$$

   defines a chain complex, the so-called mapping cone $E(\phi_\bullet)$ of $\phi_\bullet$.

   (b) Show that the inclusion $\iota : D_\bullet \to E_\bullet$ is a chain map.

   (c) Show that any commutative diagram of chain maps

   $$\begin{array}{ccc} C_\bullet & \xrightarrow{\phi_\bullet} & D_\bullet \\ \downarrow{\scriptstyle \psi_C} & & \downarrow{\scriptstyle \psi_D} \\ C'_\bullet & \xrightarrow{\phi'_\bullet} & D'_\bullet \end{array}$$

   induces a chain map of the mapping cones $E(\phi_\bullet) \to E(\phi'_\bullet)$.

   (d) Show that the chain map $\phi_\bullet : C_\bullet \to D_\bullet$ induces a long exact sequence

   $$\cdots \to H_n(C) \to H_n(D) \to H_n(E(\phi)) \to H_{n-1}(C) \to \cdots$$

   in homology.

5. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of abelian categories with enough projectives. Suppose that we are given an exact sequence

   $$0 \to K \to P_{n-1} \to \ldots \to P_1 \to P_0 \to M \to 0$$

   in $\mathcal{C}$, where all modules P_i are projective.

(a) Show that for all $i > n$ the identity

$$L_iF(M) \cong L_{i-n}F(K)$$

holds.

Hint:

There are two possible solutions: either use a projective resolution of K to construct a projective resolution of M. Or first consider the case $n = 1$ and then use induction.

(b) Conclude that $L_1F = 0$ implies $L_iF = 0$ for all i.