Exercise Sheet 10, Advanced Algebra, Summer Semester 2017. To be discussed on Thursday 29.6.17

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1. Let $\zeta \in \mathbb{C}$ be a primitive n-th root of unity.

   (a) Show that $\sum_{i=0}^{n-1} \zeta^i = 0$. What happens when ζ is not a primitive root of unity?

   (b) Show how one can receive the above result from the orthogonality of characters. Hint: use the cyclic group of order n C_n.

   (c) *By Wedderburn Theorem we can write $\mathbb{Q}C_n \cong \prod_i L_i$ where L_i are matrix algebras over division rings. Since the ring is commutative, L_i are actually fields. What fields will we get in this decomposition?

2. Let $G = Q_8$ be the quaternion group of order 8. This is the group formed by the elements $\{\pm 1, \pm i, \pm j, \pm k\}$ inside the quaternion algebra. The multiplication in this group is given by the following rules: -1 is central, $(\pm i)^2 = (\pm j)^2 = (\pm k)^2 = -1$, and $ij = k, jk = i, ki = j$, $ji = -k, ik = -j, kj = -i$. Write down explicitly the conjugacy classes in Q_8 and the character table.

3. Do the same for the Dihedral group D_8. Compare with the results of the previous exercise.

4. Let G be a finite group, and let X be a finite G-set (that is- a set upon which G acts). We define a complex representation V of G in the following way: V has a basis $\{e_x\}_{x\in X}$, and the action of $g \in G$ on the basis elements is given by $g \cdot e_x = e_{gx}$. Denote the character of this representation by χ.

   (a) Show that $\chi(g)$ is the number of fixed points of g in the action on X.

   (b) Show that $(\chi, 1)$ is the number of G-orbits in X.

5. Let G be a finite group, and let V be a finite dimensional complex representation of G with character χ.

   (a) Prove that the map $T : v \mapsto \frac{1}{|G|}\sum_{g\in G} gv$ is a projection of V onto V^G (that is: $T^2 = T$. The space V^G is the subspace of G-invariant vectors in V). Conclude that $\frac{1}{|G|}\sum_{g\in G}\chi(g)$ is equal to $dimV^G$ (hint: if T is a projection, what can you say about $Tr(T)$?)

(b) If U is another G-representation with character ψ, prove that the character of $Hom(U,V)$ with the diagonal action $(g \cdot f)(u) = gf(g^{-1}u)$ is given by $g \mapsto \psi(g^{-1})\chi(g)$.

(c) Re-prove the character orthogonality formula.