Exercise Sheet 9, Advanced Algebra, Summer Semester 2017. To be discussed on Thursday 22.6.17

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1. Let D be a division ring, and let M be a D-module. We will show here that M is free. Let

$$X = \{Y \subseteq M | Y \text{ is linearly independent over } D\}.$$

Use Zorn's Lemma to prove that X has a maximal element B, and show that B is a basis for M.

2. Let R be a ring, and let $A = M_n(R)$. We will show here that studying modules over A is "as difficult" as studying modules over R. Let $\mathcal{C} = Mod - R$ and $\mathcal{D} = Mod - A$. For every R-module M we write

$$F(M) = M^{\oplus n} = M \oplus M \oplus \cdots \oplus M.$$

For every A-module N we write

$$G(N) = Hom_A(R^n, N).$$

(a) Show that $F(M)$ is an A-module for every R-module M, and that F defines a functor from $\mathcal{C}$ to $\mathcal{D}$.

(b) Show that $G(N)$ is an R-module for every A-module N, and that G defines a functor from $\mathcal{D}$ to $\mathcal{C}$ (hint: use the fact that R^n is also a left R-module).

(c) Show that $FG \cong Id_{\mathcal{D}}$ and $GF \cong Id_{\mathcal{C}}$. Conclude that F and G establish an equivalence of categories between $\mathcal{C}$ and $\mathcal{D}$.

3. Let $A_1, \ldots A_n$ be rings, and let $R = \prod_{i=1}^n A_i$ be the ring product. Prove that every R-module M can be written uniquely as the direct product $M = \prod_{i=1}^n M_i$ where M_i is an A_i-module.

4. Let $G = S_3$. In this exercise we will find all the irreducible representations of $\mathbb{C}G$.

(a) Consider the representation $V = \mathbb{C}^3$, upon which G acts by permutation of the coordinates:

$$g \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_{g^{-1}(1)} \\ a_{g^{-1}(2)} \\ a_{g^{-1}(3)} \end{pmatrix}$$

Show that V splits as the direct sum of an irreducible representation of dimension 2, and the trivial representation of dimension 1.

(b) Find another (non-isomorphic) irreducible representation of dimension 1.

(c) Prove by counting argument that the these are all the 3 irreducible representations of S_3.

5. The goal of this exercise will be to construct many new division rings, the so called *generalized quaternion algebras*. Let K be a field of characteristic $\neq 2$, and let $a,b \in K^{\times}$. Let $D = K\langle X,Y\rangle/(X^2-a, Y^2-b, XY+YX)$. We denote by x and Y the images of X and Y in D respectively.

   (a) Show that D has dimension 4 over K. Show that $\{1,x,y,xy\}$ is a basis for D over K.

   (b) Let $d = d_1 + d_2x + d_3y + d_4xy$. Prove that $d^2 \in K$ if and only if $d_1 = 0$.

   (c) Prove that if a is not a square in K (that is, if the equation $a = t^2$ has no solution in K), then the set of elements of the form $t^2 - s^2a$ in $K^{\times}$ forms a subgroup.

   (d) Prove that D is a division algebra if and only if the equation $r^2 - s^2a = b$ has no solutions in K (hint: use the previous exercise, and find the characteristic polynomial of an element d with $d_1 = 0$).