Exercise Sheet 8, Advanced Algebra, Summer Semester 2017. To be discussed on Thursday 15.6.17

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1. Let n be a natural number. Consider the ring $R = \mathbb{Z}/n$.

   (a) Prove that every finitely generated projective R-module is injective and vice versa (hint: use the module $Hom_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$, and use the fact that this is a cofree module).

   (b) For an integer $m|n$, consider the R-module $M = \mathbb{Z}/m$. Prove that M is injective $\iff$ M is projective $\iff$ $gcd(m, n/m) = 1$.

2. Let R be a PID. Let X, Y and Z be three R-modules. Assume that $X \oplus Z \cong Y \oplus Z$. Does it necessarily true that $X \cong Y$? What if we assume that the modules are finitely generated?

3. (a) Write $\mathbb{Z}/60$ as a direct sum of modules of the form $\mathbb{Z}/n_1 \oplus \cdots \oplus \mathbb{Z}/n_r$ with $n_1|n_2|\cdots|n_r$, and as a direct sum of modules of the form $\mathbb{Z}/p^t$ where p is a prime number (both are possible due to the structure theorem of finitely generated modules over a PID).

   (b) Consider the abelian group $A = \mathbb{Z}^2/(6,9)$. Write A as a direct sum of a free module and finite cyclic groups. Show that the free module in the direct sum is not uniquely defined. How many options for the free module do we have in this case?

4. Let $R = \mathbb{Z}/p^r$ where p is a prime number and r is a natural number.

   (a) Describe all the finitely generated R-modules.

   (b) *What can you say about the number of isomorphism classes of R-modules of cardinality p^n (where n is some natural number)? What can you say about the number of isomorphism classes of abelian groups of cardinality p^n? Is this number related to the number of conjgacy classes in the group S_n?

* means that the question is more difficult than the others.