Exercise Sheet 7, Advanced Algebra, Summer Semester 2017. To be discussed on Thursday 1.6.17

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1. An abelian category $\mathcal{C}$ is called *semisimple* if every short exact sequence in $\mathcal{C}$ splits. Show that the following conditions are equivalent:
   (a) The category $\mathcal{C}$ is semisimple.
   (b) Every object of $\mathcal{C}$ is projective.
   (c) Every object of $\mathcal{C}$ is injective.

2. Let G be a finite group and let K be a field. Consider the group algebra $R = KG$. Let V be an R-module.
   (a) Show that if V is finitely generated as an R-module then V is finite dimensional as a K-vector space.
   (b) Show that if V is an R-module then $V^* = Hom_K(V,K)$ is also an R-module, where the action of $g \in G$ on $f \in V^*$ is given by $(g \cdot f)(v) = f(g^{-1}v)$.
   (c) Show that $V \mapsto V^*$ defines an equivalence of categories $R - mod_{fg} \to R - mod_{fg}^{opp}$ (where by $R - mod_{fg}$ we mean only the finitely generated R-modules).
   (d) Show that $R^* \cong R$ as R-modules, where R is the regular R-module and R^* is defined as in (c).
   (e) Show that projective and injective objects in $R - mod_{fg}$ coincide.

3. Let R be a principal ideal domain (PID). Prove or give a counterexample:
   (a) Every quotient ring of R is again a PID.
   (b) Every subring of R is again a PID.

4. For any partially ordered set X, let $\mathcal{C}_X$ be the set X considered as a category (so in $\mathcal{C}_X$ we have exactly one morphism from x to y if and only if $x \leq y$, and zero otherwise).
   (a) Show that, for two partially ordered sets X and Y there is a one to one correspondence between functors $F : \mathcal{C}_X \to \mathcal{C}_Y$ and monotonous functions $f : X \to Y$.
   (b) Consider the functor $F : \mathcal{C}_{\mathbb{Q}} \to \mathcal{C}_{\mathbb{R}}$ which corresponds to the inclusion of $\mathbb{Q}$ in $\mathbb{R}$ ($\mathbb{Q}$ and $\mathbb{R}$ are considered here with their usual ordering). Does the functor F has a left adjoint? a right adjoint?