Exercise Sheet 6, Advanced Algebra, Summer Semester 2017. To be discussed on Thursday 18.5.17

(Ehud Meir and Christoph Schweigert)

---

1. Let Grp be the category of groups. Let Ab be the category of abelian groups. For every group G, let $Z(G)$ be the center of G (which is an abelian group). Let $[G,G]$ be the subgroup of G generated by the commutators $[g,h] := ghg^{-1}h^{-1}$ for $g,h \in G$, and let G_{ab} be the abelian group $G/[G,G]$.

   (a) Is there a functor $F : Grp \to Ab$ such that $F(G) = G_{ab}$ for every group G?

   (b) Show that there is a functor $F : Grp \to Grp$ with $F(G) = [G,G]$ for every group G. Is it true that the inclusion $[G,G] \hookrightarrow G$ determines a natural transformation $F \to Id$?

   (c) Is there a functor $F : Grp \to Ab$ such that $F(G) = Z(G)$ for every group G?
   Hint: consider the group homomorphisms $\mathbb{Z}/2 \to S_3$ $a \mapsto (12)^a$ and $S_3 \to \mathbb{Z}/2$, $\sigma \mapsto \frac{1-sign(\sigma)}{2}$

2. Let $\mathcal{C} = R-mod$ be the category of R-modules. Let $U : \mathcal{C} \to Ab$ be the forgetful functor. Show that there is a bijection between the collection $Nat(F,F)$ of all natural transformations from F to itself, and the ring R.
   Hint: Let α be such a transformation. Consider first $\alpha(R) : U(R) \to U(R)$, and show that this must be given by the action of some element $r \in R$.
   Then for an R-module M and an element $m \in M$, use the fact that there is a homomorphism of R-modules $R \to M$ which sends 1 to m.

3. A *generator* of a category $\mathcal{C}$ is an object J of $\mathcal{C}$ such that for any pair of morphisms $f \neq g : X \to Y$ there exists a morphism $h : J \to X$ such that $fh \neq gh$.

   (a) Does the category Grp have a generator?

   (b) Does the category $fin-Grp$ (of finite groups and group homomorphisms) have a generator?

4. Let Top be the category of topological spaces. Let $Dis : Set \to Top$ be the functor which assigns to a set X the topological space X with the discrete topology. Does the functor Dis have a left adjoint? a right adjoint? if so, describe them.

5. Let $F : Grp \to Set$ be the functor which sends the group G to the underlying set of the group G_{ab}. Is the functor F representable (Darstellbar)?