The functor *Dis* does not admit a left adjoint

Consider the functor $Dis : Set \to Top$ which sends each set S to the topological space S with the discrete topology. In class we have asked if this functor admits a left adjoint F, and I made a mistake of not considering topological spaces such as $\mathbb{Q}$.

Let us see now why the functor *Dis does not* admit a left adjoint. Assume that such a left adjoint F exists. Then it holds that for every topological space X and every set S there is a bijection of sets

$$Hom_{Top}(X, Dis(S)) \cong Hom_{Set}(F(X), S).$$

Consider in particular the set $S = \{0,1\}$ and the topological space

$$X = \{\frac{1}{n}\}_{n\in\mathbb{N}} \cup \{0\}$$

with the subspace topology (we think of X as a subspace of $\mathbb{R}$). If $f : X \to \{0,1\}$ is a continous function, then $f(\frac{1}{n}) = f(0)$ for a big enough n. Let us write $Y_{i,n} = \{f : X \to \{0,1\} | \forall m \geq n \quad f(\frac{1}{m}) = i\}$ where $i \in \{0,1\})$. Then it holds that $Y := Hom_{Top}(X, Dis(S)) = \bigcup_{i,n} Y_{i,n}$. Since for every i and every n the set $Y_{i,n}$ is finite, we see that Y is a countable union of finite sets, and therefore countable. On the other hand, if $F(X)$ is a finite set then $Hom_{Set}(F(X), S)$ is finite, and if X is an infinite set then $Hom_{Set}(F(X), S)$ is infinite and uncountable. This gives us a contradiction, and the functor F does not exist. Notice that we have used here the fact that the point $0 \in X$ does not have any connected neighborhood.