Exercise Sheet 5, Advanced Algebra, Summer Semester 2017. To be discussed on Thursday 11.5.17

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1. Let R be a commutative ring without zero divisors. Let I_1 and I_2 be ideals in R such that $I_1 + I_2 = R$ as R-modules.

   (a) Construct an exact sequence of R-modules

   $$0 \to I_1 \cap I_2 \to I_1 \oplus I_2 \to R \to 0$$

   and deduce an isomorphism

   $$I_1 \oplus I_2 \cong (I_1 \cap I_2) \oplus R$$

   (b) Show that the four R-modules $I_1, I_2, I_1 \cap I_2$ and R are indecomposable R-modules.

2. Let $(X, \leq)$ be a partially ordered set which we can see as a category whose objects are the elements of S and whose morphisms $\mathrm{Hom}_S(x, y)$ are the one element set, iff $x \leq y$, and the empty set else. Since there is at most one morphism between two objects, the composition of morphisms is uniquely fixed.

   Describe the families $(x_i)_{i \in I}$ for which products and coproducts are defined. Describe the products and coproducts explicitly.

3. An initial object of a category is an object $*$ such that for any object U the set $\mathrm{Hom}(*, U)$ has exactly one element.

   An terminal object of a category is an object 0 such that for any object U the set $\mathrm{Hom}(U, 0)$ has exactly one element.

   (a) Show that any two inital objects are isomorphic. Show that any two terminal objects are isomorphic.

   (b) Give an example of a category in which inital and terminal objects exist, but are not isomorphic. Given an example, in which they are isomorphic.

   (c) Consider the categories of groups, of abelian groups, of rings, of unital rings (with unit preserving morphisms), of fields and of topological spaces. Do they have an initial object? Do they have a terminal object? If so, describe these objects!

4. (a) Let R be a ring. Consider the identity functor $F := \mathrm{id}_{R-\mathrm{Mod}}$. Show that natural transformations $F \Rightarrow F$ are in bijection to elements of the center of R,

$$Z(R) := \{r \in R \mid rs = sr \quad \text{for all} \quad s \in R\}$$

(b) Define a suitable natural ring structure on the endomorphisms of the identity functor. Is is possible to chose the bijection in (a) in such a way that it is an isomorphism of rings?