Exercise Sheet 4, Advanced Algebra, Summer Semester 2017. To be discussed on Thursday 4.5.17

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1. Let P be a projective R-module.

   (a) Is there always a *free* R-module F such that the direct sum $P \oplus F$ is free?

   Hint:
   Let P' be a module such that $P \oplus P'$ is free. Consider the countable direct sum
   $$P' \oplus (P \oplus P') \oplus (P \oplus P') \dots$$

   (This trick is known as the "Eilenberg swindle".)

   (b) Can F be chosen to be a finitely generated free module? (Proof or counterexample.)

2. Consider the commutative diagram of R-modules (where R is some ring)

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \phi_4} & & \downarrow{\scriptstyle \phi_5} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

   in which both rows are assumed to be exact sequences and for which ϕ_1, ϕ_2, ϕ_4 and ϕ_5 are assumed to be isomorphisms. Show that then also ϕ_3 is an isomorphism. (This is called the five lemma.)

3. Let R be an integral domain let and M be an R-module. An element $x \in M$ is said to be *divisible*, iff $x \in \cap_{\alpha \neq 0} \alpha M$. The module M is called divisible, if all its elements are divisible.

   (a) Show that the subset $\mathrm{Div}(M)$ of divisible elements of M is a submodule of M.

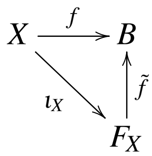
   (b) Compute $\mathrm{Div}(M/\mathrm{Div}(M))$.

   (c) Show that if M is divisible and $U \subset M$ a submodule, then the quotient module M/U is divisible.

   (d) Is any submodule of a divisible module divisible? Proof or counterexample!

4. Let R be any ring and F_X a free R-module with basis X. Since X is a subset of F, there is a natural map of sets $\iota_X : X \to F_X$.

   (a) Show that the pair (F_X, ι_X) is characterized, up to unique isomorphism, by the following property:
   For any R-module B and any map $f : X \to B$ of sets, there exists a unique morphism $\tilde{f} : F_X \to B$ of R-modules such that the following diagram commutes



   (b) Reformulate this statement as a bijection between certain sets of homomorphisms in different categories.

5. Consider the following property (C) for an R-module M:
   There is a family $(m_i)_{i\in I}$ of elements $m_i \in M$ and a family $(\Phi_i)_{i\in I}$ of elements $\Phi_i \in M^* := \mathrm{Hom}_R(M,R)$ such that:

   (1) For any $m \in M$, one has $\Phi_i(m) = 0$ for almost all $i \in I$.

   (2) For all $m \in M$, one has

$$\sum_{i\in I} \Phi_i(m) m_i = m \ .$$

   Show that a module is proejctive if and only if it has the property (C).