Exercise Sheet 3, Advanced Algebra, Summer Semester 2017. To be discussed on Thursday 27.4.17

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1. Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$.

2. Let K be a field, and let $R = M_n(K)$. Consider the K-vector spaces

$$V = \{(x_1, \ldots x_n) | x_i \in K\} \text{ and } W = \{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} | x_i \in K\}.$$

   (a) Show that V is an $K - R$ bimodule, and that W is an $R - K$-bimodule.

   (b) Prove that $V \otimes_R W \cong K$ as $K - K$-bimodules.

3. (Satz 1.4.3. im Skript) Let R be a ring and let $0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ be a short exact sequence of R-modules. Show that the following three conditions are equivalent:

   (a) There exists an R-module homomorphism $\pi : M \to M'$ such that $\pi i = Id_{M'}$.

   (b) There exists an R-module homomorphism $s : M'' \to M$ such that $ps = Id_{M''}$.

   (c) exists an R-module homomorphism $\phi : M \to M' \oplus M''$ such that $\phi i = i_1$ and $pr_2 \phi = p$ (where $i_1 : M' \to M' \oplus M''$ is given by $x \mapsto (x, 0)$ and $pr_2 : M' \oplus M'' \to M''$ is given by $(x, y) \mapsto y$).

4. Let m, n be two integers. Explain why if $gcd(m, n) = 1$ then $\mathbb{Z}/mn \cong \mathbb{Z}/m \oplus \mathbb{Z}/n$ as abelian groups. Conclude that there are submodules of free $\mathbb{Z}/mn$-modules which are not free.

5. Consider the ring $R = K[X, Y]$ where K is a field. Consider the ideal $I = (X, Y)$. Is I free as an R-module? Is I projective?