Exercise Sheet 2, Advanced Algebra, Summer Semester 2017. To be discussed on Thursday 20.4.17

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1. Let R be an integral domain. We know that in this case for every left R-module M, $Tor(M)$ is an R-submodule of M. Let $(M_\lambda)_{\lambda\in\Lambda}$ be a collection of left R-modules. Prove or give a counterexample:

   (a) $Tor(\oplus_\lambda M_\lambda) = \oplus_\lambda Tor(M_\lambda)$.

   (b) $Tor(\prod_\lambda M_\lambda) = \prod_\lambda Tor(M_\lambda)$.

2. Let R be a ring. Let M be a finitely generated left R-module, generated by k elements.

   (a) Is it true that if $N \subseteq M$ is a finitely generated submodule, then N has a generating set with at most k elements?

   (b) Is it true that every submodule $N \subseteq M$ is also finitely generated?

   (c) Let $N_1, N_2 \subseteq M$ be two finitely generated submodules. Is it true that $N_1 + N_2$ is finitely generated? Is it true that $N_1 \cap N_2$ is finitely generated?

   (d) Assume that $N_1, N_2 \subseteq M$ are two submodules. Assume also that $N_1 + N_2$ is finitely generated, and that $N_1 \cap N_2$ is finitely generated. Is it true that N_1 and N_2 are finitely generated?

3. Let R be a ring. Assume that $0 \to M \xrightarrow{i} N \xrightarrow{p} L \to 0$ is a short exact sequence of left R-modules (this means that i is injective, p is surjective, and $Ker(p) = im(i)$ as submodules of N). Assume that M and L are finitely generated. Prove that N is finitely generated.

4. Calculate $Hom_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n)$ for two integers m and n.

5. Let R be a ring, let I be a right ideal in R and let M be a left R-module. Prove that $R/I \otimes_R M \cong M/IM$ as abelian groups, where $IM = \{\sum_i x_i m_i | x_i \in I, m_i \in M\} \subseteq M$.

6. Prove that if R is a commutative ring and I and J are two ideals in R then $R/I \otimes_R R/J \cong R/(I+J)$.

7. Is it true that if R is a ring, M is a right R-module, N is a left R-module and $M \otimes_R N = 0$ then $M = 0$ or $N = 0$?