Exercise Sheet 1, Advanced Algebra, Summer Semester 2017. To be discussed on Thursday 13.4.17

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1. Let $M \subseteq \mathbb{Q}$ be a finitely generated $\mathbb{Z}$-submodule. Show that M is cyclic. Is the result still true if we replace $\mathbb{Z}$ by an integral domain R and $\mathbb{Q}$ by the ring of fractions of R? (Hint: consider $R = K[X,Y]$ where K is a field).

2. Let R be a commutative ring, and let m,n be natural numbers. Assume that $R^n \cong R^m$ as left R-modules. Prove that $n = m$. Are there also non-commutative rings for which this result holds?

3. Let R be a ring and let M be a left R-module. Is $Tor(M)$ closed under addition? Is $Tor(M)$ closed under multiplication with an element in R in case R is commutative?

4. An element $e \in R$ is called *idempotent* if $e = e^2$. An element $e \in R$ is called *central* if $er = re$ for all $r \in R$. Assume that R contains a central idempotent e, which is different from 0 and 1. Show that R is isomorphic with $R_1 \times R_2$ for two non-zero rings R_1 and R_2.

5. Let $G = S_3$, the symmetric group on three elements. Let $M = \mathbb{Q}^3$. We denote the standard basis of M over $\mathbb{Q}$ by $\{e_1, e_2, e_3\}$. We define an action of G on M by $\sigma \cdot e_i = e_{\sigma(i)}$. Show that this equips M with a structure of a $\mathbb{Q}G$-module. Describe all the submodules and quotient modules.

6. Let $T : V \to W$ be a linear surjective map of vector spaces over a field K. Show that T *splits*: that is, there exists a linear map $S : W \to V$ such that $TS = Id_W$. Is the result still true for modules over rings?