Tutorial 1.1: Combinatorial Set Theory

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ESSLLI in Ljubljana, Slovenia, August 1, 2011

I. Overview

- The König Infinity Lemma: an infinite tree with finite levels has an infinite branch.
- Ramsey's Theorem: if the edges of a complete graph on an infinite set are colored with finitely many colors, there is an infinite subset of the vertices all of whose edges are monochromatic.
- There is a graph coloring theorem for finite graphs that cannot be proved in Peano arithmetic.

II. Set theoretic trees

Set theoretic trees are partial orders $(T, <_T)$ such that for every $t \in T$, the set $T{\restriction}_t$ of predecessors of t is well-ordered, where

$$T{\restriction}_t = \{s \in T : s <_T t\}.$$

Example

Finite sequences of 0's and 1's ordered by end-extension is a tree (the *complete binary tree*) with finite levels and root the empty sequence.

Definition

A node t in a tree $(T, <_T)$ is on level $i < \omega$ if $T{\restriction}_t$ has order type i, i.e. is order isomorphic to $\{0, 1, \ldots, i-1\}$.

III. König Infinity Lemma

Theorem 1 Dénes Kőnig 1926 [3]

An infinite tree with finite levels has an infinite branch.

Definition

A *branch* is a maximal chain (totally ordered subset) of the tree.

Suppose $(T, <_T)$ is an infinite tree with finite levels. Then it must have nodes on infinitely many different levels.

III. König Infinity Lemma

There are only finitely many nodes on level 0 and every node is comparable with one of them. Let t_0 be a node on level 0 that is below nodes from infinitely many different levels.

At stage $i+1$, choose a node t_{i+1} on level i extending t_i that is below nodes from infinitely many levels.

Since this process continues for all $i < \omega$, $C = \{t_i : i < \omega\}$ is an infinite chain (all pairs comparable). Extend C to a maximal chain B in T to get an infinite branch.

IV. Ramsey's Theorem

Definition

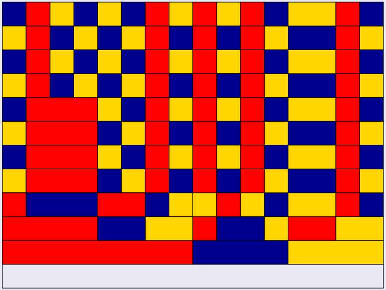
A *graph* is (V, E) where V is a set of vertices and $E \subseteq [V]^2$ is a set of edges (two element subsets of V). The *complete graph* on a set V is the graph for which $E = [V]^2$.

A *coloring* of a complete graph $(V, [V]^2)$ with colors $\{0, 1, \ldots, k-1\}$ is a function $c : [V]^2 \to \{0, 1, \ldots, k-1\}$.

IV. Ramsey's Theorem

Theorem 2 (Ramsey's Theorem for pairs [5])

For any finite coloring $c : [\omega]^2 \rightarrow \{0, 1, \ldots, k-1\}$, there is an infinite subset $U \subseteq V$ such that c is constant on $[U]^2$.



IV. Ramsey's Theorem

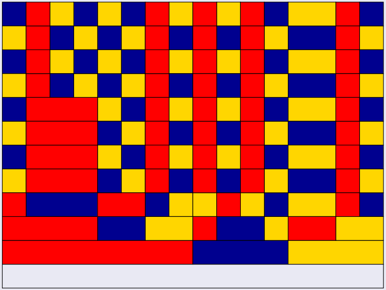
By recursion build a tree $(T, \supseteq)$ of non-empty subsets of ω indexed by k-ary sequences:

- Start with $A_\emptyset = \omega$ and let $a_0 = 0$ be the least element of $A_\emptyset$.
- If A_s has been defined and $a_s = \min A_s$ for some sequence s whose elements come from the set of colors, then for each color ℓ for which there is some $m \in A_s$ different from a_s with $f(m, a_s) = \ell$, let

$$A_{s^\frown\langle \ell \rangle} = \{b \in A_s \mid b \neq a_s \wedge f(a_s, b) = \ell\}.$$

IV. Ramsey's Theorem

- Each level of the tree is finite, since the root is the only node of level 0, and level $m + 1$ has at most k times as many nodes as level m.
- The tree has infinitely many nodes, since each $n < \omega$ is $n = a_s$ for some s which has length at most n.



IV. Ramsey's Theorem

- Apply the König Infinity Lemma to $(T, \supseteq)$ to get a branch B.
- Let $S : \omega \to \{0, 1, \ldots, k-1\}$ be $\bigcup\{s : A_s \in B\}$.
- Define $h : \omega \to \{0, 1, \ldots, k-1\}$ by $h(i) = f(a_{S\restriction_i}, a_{S\restriction_{i+1}})$.
- By the Pigeonhole Principle there is $\ell < k$ and an infinite set $H \subseteq \omega$ such that $h(i) = \ell$ for all $\ell \in H$.
- The set $\{a_{S\restriction_i} : i \in H\}$ is monochromatic of color ℓ. (Based on the Erdős proof in [2] for uncountable graphs.)

IV. Ramsey's Theorem

Theorem 3 (Infinite Ramsey's Theorem [5])

For any $r < \omega$ and finite coloring $c : [\omega]^r \to \{0, 1, \ldots, k-1\}$ of the r-tuples of ω, there is an infinite subset $U \subseteq V$ such that c is constant on $[U]^r$.

- For $r = 1$, this is the Pigeonhole Principle.
- For $r = 2$, this is Ramsey's Theorem for Pairs sketched above.
- If it is true for $r = m$, one can prove it for $r = m + 1$ by an argument similar to the one sketched, where at level $n + 1 \geq r$, we partition A_s into pieces so that for all x, y in one piece and all $a_0, a_1, \ldots, a_{m-1}$ minimal elements of $A_{t_i} \subseteq A_s$, $c(a_0, \ldots, a_{m-1}, x) = c(a_0, \ldots, a_{m-1}, y)$.
- If $H = \{a_i \mid i < \omega\}$ are the nodes along the infinite branch, then color of an $(m+1)$-element subset of H depends only on it first m elements, so we can apply the induction hypothesis.

V. Paris-Harrington Principle

Definition

The *Paris-Harrington Principle* is the statement that for all positive integers r, s, k, there is some $N < \omega$ such that for every coloring $c : [\{0, \ldots, N-1\}]^r \to \{0, 1, \ldots, k-1\}$ there is a min-size-homogeneous set X for c of size at least s.

Identify N with the set $\{0, 1, \ldots, N-1\}$ and k with $\{0, \ldots, k-1\}$.
Call a set $X \subseteq N$ *homogeneous* for $c : [N]^r \to k$
if c is constant on $[X]^r$.
X is *min-size-homogeneous* if it is homogeneous and $\min X < |X|$.

A pair $(X, [X]^r)$ is called a *complete r-uniform hypergraph*.

V. Paris-Harrington Principle

Example

If $c : [N]^2 \to 2$ is defined by $c(\{x, y\}_<) = 0$ iff y is even, then

- the set $\{1, 4\}$ is min-size-homogeneous (but not of size $\geq s = 3$); and
- the set $\{5, 6, 8, 10, 12\}$ is homogeneous and has size $5 \geq s = 3$ but is not min-size-homogeneous.

V. Paris-Harrington Principle

Theorem 4

Infinite Ramsey's Theorem implies the Paris-Harrington Principle.

Proof.

Assume to the contrary that r, s, k are finite positive integers for which the Paris-Harrington Principle fails. Let

$$T = \{c \mid (\exists N < \omega)(c : [N]^r \to k \text{ is a counter-example})\}.$$

For c, d in T, write $c \sqsubseteq d$ if $c = d{\restriction}_{\mathrm{dom}(c)}$.

Then $(T, \sqsubseteq)$ is an infinite tree with finite levels. $\square$

V. Paris-Harrington Principle

continued.

Use the König Infinity Lemma to get an infinite branch B.

Let $C = \bigcup B$. Then $C : [\omega]^r \to k$.

Apply the Infinite Ramsey's Theorem to C to get an infinite monochromatic subset H. Let $m = \min(H)$, and let X be the set of the first $m + 1$ elements of H. Then X is min-size-homogeneous for C.

Let $N > \max(X)$ be such that $C\restriction_N \in B$. Then X is min-size-homogeneous for $C\restriction_N$. This is a contradiction, so the theorem follows. $\square$

V. Paris-Harrington Principle

Theorem 5 (Paris, Harrington 1977 [4])

The Paris-Harrington Principle is not provable in Peano Arithmetic.

Multiple proofs exist. Bovykin [1] has a nice model theoretical proof that the related principle PH* is not provable in Peano Arithmetic:

> *PH*: for all positive integers r, s, k, there is some integer N such that for every coloring*
>
> $$c : [\{0, \dots, N-1\}]^r \to \{0, 1, \dots, k-1\}$$
>
> *there is a homogeneous set X for c with $|X| > s$ and $|X| > r(2^{r \min(H)} + 1)$.*

VI. References

[1] Andrey Bovykin. Brief introduction to unprovability. In *Logic Colloquium 2006*, Lect. Notes Logic. Assoc. Symbol. Logic, Chicago, IL, 38–64, 2009.

[2] P. Erdős. Some set-theoretical properties of graphs. *Revista de la Universidad Nacional de Tucumán, Serie A, Matemática y Fisica Téorica*, 3:363–367, 1942.

[3] D. Kőnig. Sur les correspondances multivoques. *Fund. Math.*, 8:114–134, 1926.

[4] J. Paris and L. Harrington. A mathematical incompleteness in Peano Arithmetic. In *Handbook of mathematical logic, Part D (Proof theory and constructive mathematics)*, North-Holland, Amsterdam, 1133–1142, 1977.

[5] F. Ramsey. On a problem of formal logic. *Proc. London Math. Soc. (2)*, 30(1):264–286, 1930.

A final remark