History and Philosophy of Infinity
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# Modal Set Theory

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# The Iterative Conception

# The Cumulative Hierarchy of Sets

$$
\begin{aligned}
V_0 &= \{x : x \text{ is an individual}\} \\
V_{\alpha+1} &= V_\alpha \cup \mathcal{P}(V_\alpha) \\
V_\lambda &= \textstyle\bigcup_{\alpha<\lambda} V_\alpha \text{ if } \lambda \text{ is a limit ordinal}
\end{aligned}
$$

# The Cumulative Hierarchy of Pure Sets

$$
\begin{aligned}
V_0 &= \emptyset \\
V_{\alpha+1} &= \mathcal{P}(V_\alpha) \\
V_\lambda &= \textstyle\bigcup_{\alpha<\lambda} V_\alpha \text{ if } \lambda \text{ is a limit ordinal}
\end{aligned}
$$

# Structuralist Logic

# Implication Structures

$\mathfrak{I} = \langle S, \Rightarrow \rangle$ — $S$ is a non-empty collection; $\Rightarrow$ is a relation on $S$

1. **Reflexivity:** $A \Rightarrow A$, for all $A$ in $S$
2. **Projection:** $A_1, \ldots, A_n \Rightarrow A_k$ for $1 \leq k \leq n$
3. **Contraction:** If $A_1, A_1, \ldots, A_n \Rightarrow B$, then $A_1, \ldots, A_n \Rightarrow B$
4. **Permutation:** If $A_1, \ldots, A_n \Rightarrow B$, then $A_{f(1)}, \ldots, A_{f(n)} \Rightarrow B$, for any permutation $f$ of $\{1, 2, \ldots, n\}$
5. **Weakening:** If $A_1, \ldots, A_n \Rightarrow B$, then $A_1, \ldots, A_n, C \Rightarrow B$
6. **Cut:** If $A_1, \ldots, A_n \Rightarrow B$ and $B, B_1, \ldots, B_m \Rightarrow C$, then $A_1, \ldots, A_n, B_1, \ldots, B_m \Rightarrow C$

# Implication Relations

1. Semantic consequence
2. Syntactic deducibility
3. Subset relation
   - $\mathfrak{I} = \langle U, \Rightarrow \rangle$
   - $A \Rightarrow B$ iff $A \subseteq B$
   - $A_1, \ldots, A_n \Rightarrow B$ iff $A_1 \cap \ldots \cap A_n \subseteq B$

# Structural Modals

$$\mathfrak{I} = \langle S, \Rightarrow \rangle$$

A modal operator $\phi$ is a function that maps $S$ to itself, and

M1. If $A_1, \ldots, A_n \Rightarrow B$, then $\phi(A_1), \ldots, \phi(A_n) \Rightarrow \phi(B)$

M2. There are $A$ and $B$ in $S$ such that $\phi(A \vee B) \Rightarrow \phi(A) \vee \phi(B)$ fails

# Necessity

Suppose $A_1, \ldots, A_n \Rightarrow B$. Then $\Box A_1, \ldots, \Box A_n \Rightarrow \Box B$

But $\Box(A \vee B) \Rightarrow \Box A \vee \Box B$ fails.

# Power Set

$$\mathfrak{I} = \langle U, \subseteq \rangle$$

Suppose $X_1, \ldots, X_n \Rightarrow Y$.

$X_1 \cap \ldots \cap X_n \subseteq Y$

$\mathcal{P}(X_1) \cap \ldots \cap \mathcal{P}(X_n) \subseteq \mathcal{P}(Y)$

# Power Set

$\mathcal{P}(X \cup Y) \subseteq \mathcal{P}(X) \cup \mathcal{P}(Y)$ fails.

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Suppose: $\mathcal{P}(X \cup Y) \subseteq \mathcal{P}(X) \cup \mathcal{P}(Y)$

$X \not\subseteq Y$ and $Y \not\subseteq X$

$X \cup Y \in \mathcal{P}(X \cup Y)$

$X \cup Y \in \mathcal{P}(X) \cup \mathcal{P}(Y)$

$X \cup Y \in \mathcal{P}(X)$ or $X \cup Y \in \mathcal{P}(Y)$

Either $X \cup Y \subseteq X$, and so $Y \subseteq X$

Or $X \cup Y \subseteq Y$, and so $X \subseteq Y$

# Naive Modal Set Theory

# Naive Set Theory

Ext. $\forall x \forall y [\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y]$

Comp. $\exists y \forall x [x \in y \leftrightarrow \Phi]$

# Naive Comprehension

$$\exists y \forall x [x \in y \leftrightarrow \Phi]$$

$$\exists y \forall x [x \in y \leftrightarrow x \notin x]$$

$$\forall x [x \in r \leftrightarrow x \notin x]$$

$$r \in r \leftrightarrow r \notin r$$

# Modal Logic

$w \Vdash \Diamond A$ iff there exists a $v$, $wRv$, and $v \Vdash A$
$w \Vdash \Box A$ iff for all $v$ such that $wRv$, $v \Vdash A$

$w \Vdash \blacklozenge A$ iff there exists a $v$, $vRw$ and $v \Vdash A$
$w \Vdash \blacksquare A$ iff for all $v$ such that $vRw$, $v \Vdash A$

## Modal Naive Comprehension

MNC: $\Diamond\exists y\forall x[x \in y \leftrightarrow \Phi]$

$$w \Vdash \Diamond\exists y\forall x[x \in y \leftrightarrow \Phi]$$

$$wRv$$

$$v \Vdash \exists y\forall x[x \in y \leftrightarrow x \notin x]$$

$$v \Vdash \forall x[x \in r \leftrightarrow x \notin x]$$

$$v \Vdash r \in r \leftrightarrow r \notin r$$

## Bi-Modal Naive Comprehension

BMNC: $\Diamond\exists y\forall x[x \in y \leftrightarrow \blacklozenge\Phi]$

$$w \Vdash \Diamond\exists y\forall x[x \in y \leftrightarrow \blacklozenge\Phi]$$

$$wRv$$

$$v \Vdash \exists y\forall x[x \in y \leftrightarrow \blacklozenge(x \notin x)]$$

$$v \Vdash \forall x[x \in r \leftrightarrow \blacklozenge(x \notin x)]$$

$$v \Vdash r \in r \leftrightarrow \blacklozenge(r \notin r)$$

# Bi-Modal Naive Comprehension

$$v \Vdash r \in r \leftrightarrow \blacklozenge(r \notin r)$$

$v \Vdash r \notin r$.

$v \Vdash \neg\blacklozenge(r \notin r)$

$v \Vdash \blacksquare(r \in r)$

$w \Vdash r \in r$

$v \Vdash r \in r$

$v \Vdash r \in r$

$v \Vdash \blacklozenge(r \notin r)$

$w' \Vdash r \notin r$

# ZF Axioms

Empty Set: $\Diamond\exists y\forall x[x \in y \leftrightarrow \blacklozenge(x \neq x)]$

Power Set: $\Diamond\exists y\forall x[x \in y \leftrightarrow \blacklozenge(x \subseteq a)]$

Infinity: $\Diamond\exists y\forall x[x \in y \leftrightarrow \blacklozenge\Diamond(x \subseteq y)]$

# Infinity

$$w \Vdash \Diamond\exists y[\emptyset \in y \wedge \forall x(x \in y \rightarrow \{x\} \in y)]$$

$wRv$ and $\emptyset$ exists at $v$.

From BMNC, $v \Vdash \Diamond\exists y\forall x[x \in y \leftrightarrow \blacklozenge\Diamond(x \subseteq y)]$

$vRu$ and $u \Vdash \forall x[x \in i \leftrightarrow \blacklozenge\Diamond(x \subseteq i)]$

$u \Vdash \emptyset \subseteq i$

$u \Vdash \blacklozenge\Diamond(\emptyset \subseteq i)$

$u \Vdash \emptyset \in i$

# Infinity

$$w \Vdash \Diamond\exists y[\emptyset \in y \wedge \forall x(x \in y \rightarrow \{x\} \in y)]$$

$$u \Vdash z \in i$$

$$u \Vdash \{z\} \subseteq i$$

$$u \Vdash \blacklozenge\Diamond(\{z\} \subseteq i)$$

$$u \Vdash \forall x[x \in i \leftrightarrow \blacklozenge\Diamond(x \subseteq i)]$$

$$u \Vdash \{z\} \in i$$

$$u \Vdash \exists y[\emptyset \in y \wedge \forall x(x \in y \rightarrow \{x\} \in y)]$$

$$v \Vdash \Diamond\exists y[\emptyset \in y \wedge \forall x(x \in y \rightarrow \{x\} \in y)]$$

$$w \Vdash \Diamond\Diamond\exists y[\emptyset \in y \wedge \forall x(x \in y \rightarrow \{x\} \in y)]$$

$$w \Vdash \Diamond\exists y[\emptyset \in y \wedge \forall x(x \in y \rightarrow \{x\} \in y)]$$

Thank you.


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