Multiverse Conceptions and the Hyperuniverse Programme

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A glimpse of: The Hyperuniverse Programme

- Launched in Friedman-Arrigoni, [6]. Based on work by Friedman, joint work by Friedman and Arrigoni ([5], [4]) and Friedman and Honzik ([7]). Wrt to the features of the Hyperuniverse Programme, the present paper draws upon and expands on [6].
- Description of the set-theoretic multiverse.
- Investigation of philosophically justified mathematical criteria.
- Search for *new axioms*.
- Search for *new set-theoretic truths*.

Multiverse Concept

T-Multiverse

Any model M of a theory T is a universe of T. The T-multiverse is the collection of all models of T.

Non-Vacuousness of the Multiverse Concept

In any T-multiverse, there must be at least two models of T which differ from each other.

Multiverse Phenomenon

It is possible to generate (infinitely many) different universes of set theory (e.g., a *non-vacuous ZFC*-multiverse, for instance).

Hence, one step further in the development of set theory:

Multiverse Description

Set theory deals with different universes of sets. These are constructed through the methods of *forcing*, *ultrapowers*, *model-theoretic methods*, ... Set-theorists aim to describe the properties of such universes and the relationships between them.

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- The 'radical multiverse view' (Balaguer, Hamkins).
- Pluralism (Shelah).
- A 'restrictive' conception (Woodin).

FBP (Full-blooded Platonism): all conceivable mathematical universes *exist*. (Balaguer, [1], [2]).

Hamkins' stance takes up *FBP*:

> *The multiverse view is one of higher-order realism - Platonism about universes - and I defend it as a realist position asserting actual existence of the alternative set-theoretic universes into which our mathematical tools have allowed us to glimpse. ([8], p. 417)*

Balaguer's 'radical' multiverse conception, as further elaborated by Hamkins:

> *The background idea of the multiverse, of course, is that there should be a large collection of universes, each a model of (some kind of) set theory. There seems to be no reason to restrict inclusion only to ZFC models, as we can include models of weaker theories ZF, ZF^-, KP and so on, perhaps even down to second order number theory, as this is set-theoretic in a sense. ([8], p. 436)*

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- FBP might imply unwarranted *ontological inflation*. Arithmetical statements are not changed through *forcing*, hence an object such as ω, for instance, does not vary in models obtained through forcing. However, FBP, quite implausibly, requires that each model relates to a different set concept, including different concepts of ω.

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- Concerns about multiverse-membership: are ill-founded models of *ZFC* universes of set theory?
- On Hamkins' view, problems such as the Continuum Problem are settled, by simply asserting that CH is true in some universes and false in others. This is a sterile point of view, which leads to no progress in our understanding of set-theoretic truth.

Shelah's Pluralism

Claim 1: there are different extensions of ZFC, each with its own collection of models, none of which is better than any other.

Claim 2: there are no preferred such extensions of ZFC. Some axioms may be fruitful in terms of their consequences (in that case, Shelah calls them *semi-axioms*), but that does not imply that they are 'better' than other axioms.

My mental picture is that we have many possible set theories, all conforming to ZFC. I do not feel a "universe of ZFC" is like the Sun", it is rather like "a human being" or "a human being of some fixed nationality". ([10], p. 211)

Generally, I do not think that the fact that a statement solves everything really nicely, even deeply, even being the best semi-axiom (if there is such a thing, which I doubt), is a sufficient reason to say that it is a "true" axiom. In particular, I do not find it compelling at all to see it as true. ([10], p. 212)

Problems with Shelah's conception:

- Same as with any *formalistic* conception (why did we pick up the *ZFC* axioms and why do we stop with those?)
- Task of establishing new set-theoretic truths is barred.
- Notion of *semi-axiom* might be too vague.

Woodin's position, as expressed in [11], [13], [12]:

- **Multiverse** = the collection of all $V_\alpha^{\mathbb{B}}$ = all *Boolean-valued* universes, which are generated through *set-forcing* = **set-generic multiverse**.
- Fix *Multiverse Laws* (ML) = (Set-generic Multiverse Laws), in analogy with Tarski's notion of truth. (see [13])
- Through Ω-conjecture and acceptance of class-many Woodin cardinals, ML are violated.
- As a consequence, the set-generic multiverse conception is flawed.

Discussion

- Any restriction of the multiverse to only one method of model-construction (*set-forcing*) is untenable [in our view, this is CORRECT].
- 'Set-generic multiverse = Multiverse' (A), hence the multiverse concept is flawed. [in our opinion, this is NOT CORRECT]. Acceptance of (A) leads to a misrepresentation of the multiverse concept.

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- In *ZFC*, one can prove that there are forcing extensions of *countable models*.
- The Hyperuniverse can be put to work with the main goal of searching for *new axioms*.

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- Mathematical Criteria (MC) = higher-order set-theoretic statements (see, e.g., IMH, IMH♯, Refl, etc.), universes which satisfy them.
- Axioms = first-order consequences of MC, relevant first-order set-theoretic statements which hold in all universes where MC hold.

1st Step: Philosophical Principles

These are *theoretical desiderata* concerning the nature of V. For instance, we may want to accept MAXIMISE, insofar as we wish V to be as rich as possible. We may want to opt for UNIFORMITY insofar as we want V to have the same structure at different levels (= reflect to rank initial segments), etc. Ideally, we would also like to investigate the *epistemic* import of these principles (i.e., are there any *meta-principles* justifying them?)

2nd Step: Mathematical Criteria

These are, usually, higher-order mathematical statements, which turn PP into MC. E.g.:

IMH (Inner Model Hypothesis) (expressing *power-set maximality*): If a parameter-free sentence holds in some outer model of v (i.e., in some universe w containing v with the same ordinals as v), then it holds in some inner model of v (i.e., in some universe v_0 contained in v with the same ordinals as v).

Note: v denotes a picture of the "real universe" V, as depicted by a countable transitive model of *ZFC* (i.e. element of the Hyperuniverse).

3rd Step: New Axioms

Once universes satisfying specific MC have been found, statements which hold in all such universes may be declared *new axioms*. For instance, SCH (Singular Cardinal Hypothesis) holds in all universes where *IMH* holds.

One further example:

SIMH (Strong IMH), Special Case (power-set maximality with parameters ω_1 and ω_2): If a sentence with parameters ω_1 and ω_2 holds in some outer model of v with the same ω_1 and ω_2 as v, then it holds in some inner model of v with the same ω_1 and ω_2 as v.

First-order consequences: Those of IMH together with *the negation* of CH, a solution to the Continuum Problem!

Mathematical Goals

One example:

IMH♯ (Friedman-Honzik): The IMH for universes with the maximum degree of "vertical reflection" (= ordinal maximality), the "♯-generated" universes.

SIMH♯ (special case): The IMH♯ with parameters ω_1 and ω_2, as with the SIMH.

The SIMH♯ also gives the negation of CH, but, unlike the IMH, IMH♯ is consistent with (but does not imply) the existence of large cardinals.

Conjecture. SIMH♯ is consistent.

Mathematical Goals (Cont'd)

to propose **new axioms**, *in the wake of, among others, Gödel's suggestions, to extend ZFC by adding consequences of mathematical criteria which are based on justifiable philosophical principles.*

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- Our project characterises itself for attempting to tie the search for new axioms to *intrinsic evidence*, that is evidence provided by the nature of set-theoretic principles and concepts, as resting upon *philosophical justifications*.

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- Hence, one major philosophical goal of the programme is to show how the process of justification of new axioms works, in view of two main concerns: 1. the existence and inevitability of the multiverse phenomenon. 2. the necessity of finding intrinsic evidence for the acceptance of axioms.

Thanks for your attention!


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