On Dedekind's Explanation of the Finite in Terms of the Infinite

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Introduction

- Prominent in discussions of the mathematical infinite is typically: *Georg Cantor*; earlier also *Bolzano*, *Galilei*, all the way back to *Aristotle*.
- One highly influential authority: *David Hilbert*, with his well-known remark that "no one shall drive us from the paradise that Cantor created for us" (1926).
- I want to highlight another seminal figure: *Richard Dedekind*.
- A first remark guiding me is by *Ernst Zermelo*, who wrote that modern set theory was "created by Cantor and Dedekind" (1908).
- Also Hilbert, who was fascinated by Dedekind's and Frege's attempts to "explain the finite in terms of the infinite" (1922), even though he took them to have failed.
- Finally, cf. Akihiro Kanamori: the actual infinite first "entered [mainstream] mathematics in Dedekind's work" (2012), already in the 1850s.
- Claim: Dedekind was *as important as Cantor* for the acceptance of the infinite in *mathematical practice*, in some respects *more so*. (But: less "drama".)
- Three aspects and, thus, parts of my talk:

  PART I: Dedekind's contributions to the rise of set theory

  PART II: His use of infinite sets in mathematics more generally

  PART III: The issue of "explanation" as part of "mathematical practice"

Introduction (continued)

Relevant works by Dedekind:

1872: *Continuity and Irrational Numbers*
1888: *The Nature and Meaning of Numbers*
(Cf. 1930/32: *Gesammelte Mathematische Werke, Vols. I-III*)

But also:

1872-1899: Some meetings, and intermittent correspondence, with Cantor
1860s-90s: Supplements to Dirichlet's *Lectures on Number Theory*; also his corresponding work (with H. Weber) in algebraic geometry (function fields)
1855-1858: Early work on algebra, including Galois theory (lecture notes).

Besides Zermelo's and Hilbert's remarks, I am building on the following:

- José Ferreirós: *Labyrinth of Thought* (1999/2010)
- Akihiro Kanamori: "In Praise of Replacement" (BSL, 2012)
- But also, recent work on mathematical explanation (cf. my "Dedekind, Structural Reasoning, and Mathematical Understanding", 2009)
- As more general background, cf. my survey "Dedekind's Contributions to the Foundations of Mathematics" (SEP, 2008/2011)

PART I: Dedekind's Contributions to Set Theory (Reminder)

- 1872 booklet (on continuity and irrational numbers):
  - He starts with ***Q***, seen as an infinite set—actual infinity, contra Aristotle
  - Use of cuts (infinite subsets of ***Q***) to define continuity and introduce the elements of ***R***
- 1888 booklet (but much already in early drafts from 1870s):
  - Use of a general theory of sets ("Systeme") and functions ("Abbildungen"), both understood extensionally, as a foundational framework—for ***N***, then for ***Z***, ***Q***, ***R*** etc.
  - Explicit definition of sets as (Dedekind-)"infinite"—very bold, turning a "paradox" (Galilei) into a definition (characteristic property of infinite sets). (Here also: implicit use of AC.)
  - Dedekind-Peano axioms for ***N***, via "simply infinite systems"—acknowledged by Peano.
  - Systematic justification of definitions by recursion and proofs by induction, via the notion of "chain" (a set closed under a given function)—later generalized by Zermelo and von Neumann.
  - Famous categoricity result for simply infinite sets (all isomorphic).
  - The "construction" of a simply infinite set (cf. Bolzano, put in problematic "psychologistic" language)—later the acknowledged basis for Zermelo's axiom of infinity.
- Correspondence with Cantor (from 1872 on):
  - Proof that the set of algebraic numbers, not just ***Q***, is countable—part of the inspiration for Cantor's study of the cardinality of ***R*** (non-countability discovered in 1873).
  - Proof of the Cantor-Bernstein equivalence theorem, again via Dedekind's theory of chains.

All of this became a *standard* and *integral* part of ZFC—thus Zermelo's remark.

Then again: Use of a *naïve* approach to sets, subject to Russell's antinomy; and *no basic axioms* formulated explicitly (cf. Frege's criticism, Zermelo's work).

PART II: Dedekind's Novel Uses of Infinity (More Generally)

- In the more foundational works (1970s-80s):
  - ***Q*,** then also ***R*** and ***N***, as infinite sets—or rather, as infinite relational systems (ordered etc.).
  - Real numbers as infinite sets of rationals (implicit use of, essentially, the power set axiom).
  - Construction of ***Z***, ***Q*** in terms of infinite equivalence classes or pairs in Dedekind's *Nachlass*.
- In algebraic number theory (from 1870s on):
  - Arbitrary sub-fields, as well as corresponding sub-rings, of ***C***.
  - Ideals introduced as infinite sets (subsets of rings closed under certain operations).
  - Similarly for other (often infinite) relational systems, e.g., modules, later lattices, etc.
- In algebra (already in the 1850s):
  - Quotient constructions for modular arithmetic: actually infinite residue classes treated as unitary mathematical objects here (unlike, e.g., in Gauss who works with residues directly).
  - Important: ***Z***[*x*], the ring of polynomials with integer coefficients (whose roots are algebraic numbers). Mod *p*: a class consisting of infinitely many infinite equivalence classes.
  - In Dedekind's own words: "[T]he whole system of infinitely many functions of a variable congruent to each other modulo *p* behaves here like a single concrete number in number theory. […] The system of infinitely many incongruent classes—infinitely many, since the degree may grow indefinitely—corresponds to the series of whole numbers in number theory." (Dedekind 1930/32, Vol. 1, pp. 46-47, as quoted in Kanamori 2012, p. 49.)

PART II: Dedekind's Uses of Infinity (Further Analyzed)

- Basic observations:
  - In some of these constructions, one can avoid the actual infinite easily (cf. $\boldsymbol{Z}$, $\boldsymbol{Q}$, also $\boldsymbol{Z}_p$).
  - But in other cases, the use of the actual infinite is unavoidable and essential—e.g., real numbers as cuts, ideals as infinite sets, and certain groups.
  - Thus Kanamori's remark (concerning the case $\boldsymbol{Z}[x]$): "One can arguably date the entry of the actual infinite into mathematics here [i.e., in the 1850s], in the sense of infinite totalities serving as unitary objects within an infinite mathematical system" (pp. 49-50).
- Towards "explanation":
  - In Dedekind's corresponding writings, one can find very modern looking theorems, especially homomorphism and isomorphism theorems. (Example: Given a group homomorphism of G onto H, with kernel K, we have $G/K \cong H$.)
  - They are part of an emerging, very general methodology, where we study relational systems (finite or infinite sets with certain functions and relations defined on them) and various structure-preserving mappings between them.
  - It is an infinitary, non-constructive, and "structuralist" methodology (cf. Reck 2009); often people talk about "abstract" mathematics in this connection ("abstract algebra" etc.). Both model theory and category theory are outgrowths of it.
  - Here: not (always) an issue of "foundations", but of "methodology", "reasoning style", etc.

PART III: Explanation and Mathematical Practice

- Cantor:
  - An alternative construction of ***R***, via Cauchy sequences (related to Weierstrass etc.).
  - Cantor's theory of transfinite cardinal and ordinal numbers etc.; leading to the General Continuum Hypothesis and other questions central to higher set theory.
  - Moreover, Cantor had a relatively sophisticated response to the set-theoretic antinomies.
  - In addition, there was the beginning of descriptive set theory (point sets etc.).
  - All of it was, at least initially, an outgrowth of Cantor's work in analysis (the study of real-valued functions with infinitely many singularities)—connected to mainstream mathematics.
- Dedekind:
  - Important *particular contributions* to the rise of set theory as well, i.e., some central results.
  - The *systematic* use of infinite sets, and corresponding functions, as a *foundational framework*, for studying ***N***, ***Z***, ***Q***, ***R***, ***C***, recursive processes, sets more generally, etc.
  - *Beyond* that: steps towards "abstract algebra", studies in algebraic number theory, algebraic geometry, etc.—picked up later by Noether, Bourbaki, etc.
- Thus:
  - With respect to a *foundational* perspective, Dedekind was *as important as Cantor*. (Unlike Dedekind and Frege, Cantor was initially not very interested in foundational issues.)
  - This is not to deny the importance of Cantor's unique contributions (transfinite numbers, GCH, descriptive set theory, etc.), which had a huge influence on *higher set theory*.
  - Then again, with respect to *mainstream* mathematics Dedekind's influence may have been *more pervasive than Cantor's*—and in ways that involve the infinite systematically.

PART III: Mathematical Explanation and the Infinite

- Back to Hilbert on "explaining the finite in terms of the infinite":
  - Dedekind's (and Frege's) attempts to "explain" ***N*** within a basic framework of infinite sets and functions—characterize, precisely and completely, the *first important infinity* in mathematics.
  - In Dedekind (unlike Frege), part of an encompassing methodology meant to "explain" ***R***—the *second crucial infinity*—but *also* divisibility in arbitrary subfields of ***C***, function fields, etc.
- On "explanation" in mathematics more generally:
  - A slippery notion that philosophers of mathematics have only started to address (M. Steiner, P. Kitcher, P. Mancosu, etc.), partly borrowing from philosophy of science; but no agreement.
  - Still, mathematicians often try to "account for", "make intelligible", "comprehend", "understand", etc. mathematical phenomena—an important part of "mathematical practice".
  - One can distinguish different "methodologies", "reasoning styles", etc. (H. Stein, I. Hacking, etc.); related to, but not identical, with "foundations" (derivability, truth, existence, etc.).
  - Partly: reasoning from the "right concepts"; identifying "structural properties"; systematic "variation of cases" (cf. group and number theory), analogous to "mechanistic explanation".
- Crucial for my purposes:
  - Without the huge success of the "abstract", "conceptual", or "structural" explanation style that one finds first in Dedekind, the infinite wouldn't be so *entrenched* in mathematical practice.
  - Even for many people not interested in, or suspicious of, axiomatic set theory, foundational studies, etc., giving up those uses of the actual infinite in mathematics would be a *big loss*.