Infinity and a critical view of logic

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I. What is the critical view of logic? Contrast with views attributed to Frege.

II. Brouwer and his counterexamples

1. Consider the proposition that the points of the continuum form an ordered point species, i.e for every such point r, either r > 0 or r = 0 or r < 0. Let k be the least number m, if it exists, such that

> the segment $d_m\ d_{m+1}, \ldots d_{m+9}$ of the decimal expansion of π forms the sequence 0123456789. Further, let $c_n = (-\frac{1}{2})^k$ if $\nu \geq k$, otherwise let $c_n = (-\frac{1}{2})^m$; then the infinite sequence $c_1, c_2, c_3, \ldots$ defines a real number r for which none of the conditions r = 0, r > 0, or r < 0 holds.[1]

In other words, r = 0 if there is no sequence 0123456789 in the decimal expansion of π, = $(-1/2)^k$ if k is the least m initiating such a sequence, so that r is positive or negative according as k is even or odd.

2. A *fleeing property* is a decidable property for which

> … one cannot calculate a particular number that has the property, nor can one prove the absurdity of the property for all natural numbers. We define the

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[1] "Über die Bedeutung des Satzes vom ausgeschlossenen Dritten in der Mathematik, insbesondere in der Funktionentheorie," *Journal für die reine und angewandte Mathematik* 154 (1924), 1-7, p. 3; translation from Jean van Heijenoort (ed.), *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931* (Cambridge, Mass.: Harvard University Press, 1967), p. 337, notation slightly modified.

> critical number [*Lösungszahl*] λ_f of a fleeing property f as the (hypothetical) smallest natural number that possesses the property; we further define an up number and a down number of *f* as a number that is, respectively, not smaller and smaller than the critical number. It is immediately clear that for an arbitrary fleeing property each natural number can be recognized to be either an up number or a down number and that in the first case the property loses its character as a fleeing property.[2]

III. Hermann Weyl

1 Is there a natural number with a certain (decidable) property *P*?

> Only the finding *that has actually occurred* of a determinate number with the property *P* can give a justification for the answer "Yes," and — since I cannot run a test through all numbers — only the insight, that it lies in the essence of number to have the property ¬*P*, can give a justification for the answer "No"; Even for God no other ground for decision is available. *But these two possibilities do not stand to one another as assertion and negation*.[3]

But if one runs through the numbers, either the process will break off with the discovery of a number possessing *P* or not: "It is either so or not so, without change and wavering and without a third possibility."[4]

2. An existential statement is not a judgment in the proper sense but a "judgment abstract" obtained from a judgment about a particular instance.

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[2] "Mathematik,Wissenschaft, und Sprache," *Monatshefte für Mathematik und Physik* 36 (1929), 153-164, p. 161; translation from Paolo Mancosu (ed.) *From Brouwer to Hilbert* (Oxford University Press, 1998), p. 51, slightly modified.

[3] "Über die neue Grundlagenkrise," p. 54, translation from Mancosu, op. cit., p. 97, modified.

[4] Ibid.

> If knowledge is a precious treasure, then the judgment abstract is a piece of paper indicating the presence of a treasure, without revealing at which place. Its only value can be to drive me to look for the treasure. The piece of paper is worthless as long as it is not realized by an underlying actual judgment like "2 is an even number."[5]

3. Weyl's attitude toward intuitionism in 1927

> Mathematics with Brouwer gains its highest intuitive clarity. He succeeds in developing the beginnings of analysis in a natural manner, all the time preserving the contact with intuition much more closely than had been done before. It cannot be denied, however, that in advancing to higher and more general theories the inapplicability of the simple laws of classical logic eventually results in an almost unbearable awkwardness. And the mathematician watches with pain the larger part of his towering edifice which he believed to have been built with concrete blocks dissolve into mist before his eyes.[6]

4. Weyl on Brouwer in 1946

> Brouwer made it clear, as I think beyond any doubt, that there is no evidence supporting the belief in the existential character of the totality of all natural numbers, and hence the principle of excluded middle in the form "Either there is a number of the given property γ, or all numbers have the property $\neg\gamma$" is without foundation. … Brouwer opened our eyes and made us see how far

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[5] Ibid., p. 54, trans. pp. 97-98.

[6] *Philosophie der Mathematik und Naturwissenschaft* (Munich: Oldenburg, 1927), p. 44, translation from the English version, *Philosophy of Mathematics and Natural Science* (Princeton University Press, 1949), p. 54.

> classical mathematics … goes beyond such statements as can claim real meaning and truth founded on evidence.[7]

IV. Hilbert

1. Hilbert on quantification over an infinite domain

> But in mathematics these equivalences [of ¬ ∀xAx and ∃x¬Ax and of ¬∃xAx and ∀x¬Ax] are customarily assumed, without further proof, to be valid for infinitely many individuals as well; and with this step we leave the domain of the finite and enter the domain of transfinite modes of inference. If we were consistently and blithely to apply to infinite totalities procedures that are admissible in the finite case, then we would open the floodgates of error. This is the same source of mistakes that we are familiar with from analysis. In analysis, we are allowed to extend theorems that are valid for finite sums and products to infinite sums and products only if a special investigation of convergence guarantees the inference; similarly, here we may not treat the infinite sums and products
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> $$A1 \wedge A2 \wedge A3 \ldots$$
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> $$A1 \vee A2 \vee A3 \ldots$$
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> As though they were finite, unless the proof theory we are about to discuss permits such a treatment.[8]

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[7] "Mathematics and Logic: A brief survey serving as a preface to a review of 'The Philosophy of Bertand Russell'," *American Mathematical Monthly* 53 (1946), 2-13, pp. 3, 7. Weyl explains "existential" as abbreviating his older formulation "a closed realm of things existing in themselves."

[8] "Die logischen Grundlagen der Mathematik," *Mathematische Annalen* 88 (1923), 151-165, p. 155, translation from William Ewald (ed.), *From Kant to Hilbert* (Oxford: Clarendon Press, 1996), II, 1139-1140.

2. In Euclid's proof of the infinity of the prime numbers, it is quite in accord with the finitary standpoint to prove that for every prime *p*, there is a prime between *p* + 1 and *p*! + 1.

> … and this leads us to formulate a proposition that expresses only a part of Euclid's proposition, namely: there exists a prime number that is > *p*. So far as content is concerned, this is a much weaker assertion, stating only a part of Euclid's proposition; nevertheless, no matter how harmless the transition appears to be, there is a leap into the transfinite when this partial proposition, taken out of the context above, is stated as an independent assertion.[9]

3. General statements from a finitary point of view

> In general, from the finitist point of view an existential proposition of the form "There exists a number having this or that property" has meaning only as a partial proposition, that is, as part of a proposition that is more precisely determined but whose exact content is inessential for many applications. … In like manner, we come upon a transfinite proposition when we negate a universal assertion, that is, one that extends to arbitrary numerals. So, for example, the proposition that, if *a* is a numeral, we must always have
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> $$a + 1 = 1 + a$$
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> is from the finitist point of view incapable of being negated.[10]

V. A more contemporary point of view

1. The entanglement of logic and mathematics

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[9] "Über das Unendliche," *Mathematische Annalen* 95 (1926), 161-190, p. 172, translation from van Heijenoort, p. 378.

[10] Ibid., p. 173, trans. p. 378.

2. The case of second-order logic and its entanglement with set theory

3. A question: Do we understand the application of logic to the higher infinite better than we understand the higher infinite itself?