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Gödel's Argument for Cantor's Cardinals

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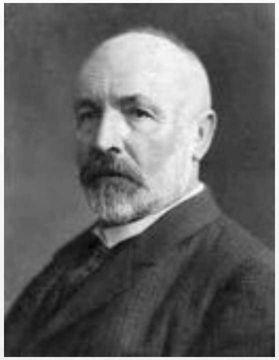
The Hume–Cantor Principle: If there is a 1-1 correspondence between two collections, then they are equal in size



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The Part–Whole Principle: If a collection A is a properly included in a collection B, then A is smaller than B

www.glogster.com

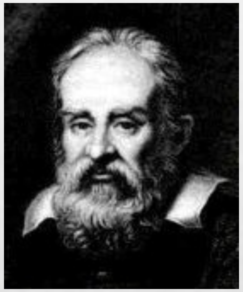
conflicting intuitions

The whole numbers can be mapped 1-1 to their squares

- So they're equal in number

Yet the whole numbers properly include their squares

- So there are more whole numbers than squares



apod.nasa.gov

Galileo: So infinite collections are incomparable

Leibniz and Bolzano: Part–Whole is undeniable so Hume–Cantor is false

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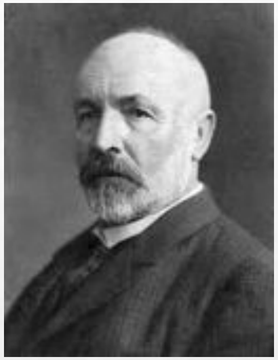
galileo's paradox

Today commonly taken for granted that Galileo, Leibniz, and Bolzano were *mistaken*

- Cantor's "power" is the *uniquely correct* concept of "how many"



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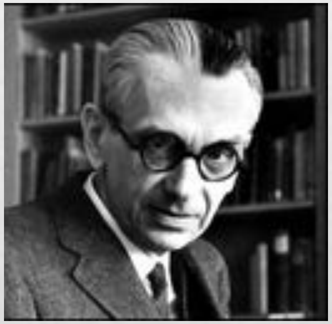
Gödel gave one of the few *arguments* for this in "What is Cantor's Continuum Problem?" (1947)

- (Others?)
- Apparently meant as an uncontroversial example to soften us up for his more radical realist views



www.nassauchurch.org

the cantorian hegemony

- MW Parker (2009), “Philosophical Method and Galileo’s Paradox of Infinity”

  in *New Perspectives on Mathematical Practices: Essays in Philosophy and History of Mathematics,* Bart van Kerkhove, ed.

  - Also in PhilSci Archive

- MW Parker (forthcoming), “Set Size and the Part–Whole Principle”, *Review of Symbolic Logic*
  - Shorter, more informal version on PhilPapers

previous criticism

‘(Part–Whole & ~ Hume–Cantor)’ is consistent with ZFC

- Not surprising; ZFC says nothing about “sizes”!

Benci, Di Nasso, and Forti’s “Numerosities”

- Satisfy Part–Whole
- Have the same 1^{st}-order algebraic and ordering properties as the integers
  (a discretely ordered semi-ring)
- Are total over the integers, the ordinals, point sets
- Exist if AC and CH (or Martin’s Axiom) hold

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euclidean theories of size

Gödel’s argument not supposed to show Part–Whole *inconsistent* (or inconsistent with ZFC)

- Supposed to show it *false*
- For Gödel, truth ≠ consistency

But to show it false, must show it inconsistent with *something*, namely true premises

So what are his premises? What’s the argument?

do numerosities refute gödel?

[Premise 2] If there is a 1-1 correspondence between two sets A and B (of changeable objects of the space-time world), it is "theoretically" possible to change the properties and relations of each element of A into those of the corresponding element of B.

[Premise 3] If the properties and relations of the elements of A are changed into those of the corresponding elements of B, then A is thus made completely indistinguishable from B.

∴ [Lemma 2] If there is a 1-1 correspondence between two sets A and B of changeable elements of the space-time world, it is "theoretically" possible to change the properties and mutual relations of the elements of A so that it has the same cardinal number as B.

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gödel's argument pt. 1

gödel’s argument pt. 2

[Lemma 2] If there is a 1-1 correspondence between two sets A and B of changeable elements of the space-time world, it is “theoretically” possible to change the properties and mutual relations of the elements of A so that it has the same cardinal number as B.

[Premise 1] We want number to have the property that the number of objects belonging to a class does not change if, “leaving the objects the same”, one changes their properties or mutual relations.

∴ [Lemma 1] Two sets of changeable objects of the space-time world have the same cardinal number if their elements can be brought into a one-to-one correspondence.

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[Lemma 1] Two sets of *changeable objects of the space-time world* have the same cardinal number if their elements can be brought into a one-to-one correspondence.

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[Premise 4] A definition of the concept of "number" that depends on the *kind* of objects that are numbered would be unsatisfactory.

∴ [Conclusion] Cantor's definition of infinite numbers is the only manner of extending the concept of number to infinite sets.

gödel's argument pt. 3

[Premise 2] If there is a 1-1 correspondence between two sets A and B (of changeable objects of the space-time world), it is *"theoretically" possible* to change the properties and relations of each element of A into those of the corresponding element

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'Theoretically' can mean

- deductively rather than empirically known
- according to a generally accepted theory
- so far as logic alone dictates (but not really)

theoretically possible??

theoretically possible??

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- Suppose the elements of one set are mass points and those of another are *systems of two* mass points
  - Can a system of two mass points be made to resemble a single mass point or vice versa, even “theoretically”?

  (Mass points are Gödel’s example of “changeable objects of the spacetime world”, but he does not consider systems of two mass points)

- Is it theoretically possible to transform *infinitely many* physical objects?

[Premise 1] We want number to have the property that the number of objects belonging to a class does not change if, "leaving the objects the same", one changes their properties or mutual relations.

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What does "leaving the objects the same" mean?

- Not changing the number of them?

  Circular.

- Never adding or removing one?

  False: In some cases we can change their properties and mutual relations so that one splits or two fuse, and then we ***do*** want the number to change.

(Anyway, ***why*** would we "want" number to have this property? Because it's true or because it has some other *practical* value?)

"leaving the objects the same"

[Premise 4] A definition of the concept of "number" that depends on the *kind* of objects that are numbered would be unsatisfactory.

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Quine: "No entity without identity"

- On this view, the way we count partly *defines* the kind of object

Why not be pluralists?

- Use Cantor's Principle where 1-1 correspondence is most important
- Use Part–Whole where subset relations are most important

This is what we actually do—even Gödel!

kind dependence

[Premise 3] If the properties and relations of the elements of A are changed into those of the corresponding elements of B, then A is thus made completely indistinguishable from B.

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This means *intrinsic* properties and *internal* relations, e.g.,

- Colors
- Distribution in space

But no: A and B might still be distinguished by their relations to *each other* or to *other* things

- Location
- Subset relation

"Euclidean" (Part–Whole) notions of set size imply these

indistinguishable?

[Premise 2] If there is a 1-1 correspondence between two sets A and B (of changeable objects of the space-time world), it is "theoretically" possible to change the properties and relations of each element of A into those of the corresponding element of B.

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∴ [Lemma 2] If there is a 1-1 correspondence between two sets A and B of changeable elements of the space-time world, it is "theoretically" possible to change the properties and mutual relations of the elements of A so that it has the same cardinal number as B.

a tacit premise

[Premise 2] If there is a 1-1 correspondence between two sets A and B (of changeable objects of the space-time world), it is “theoretically” possible to change the properties and relations of each element of A into those of the corresponding element of B.

[Premise 3] If the properties and relations of the elements of A are changed into those of the corresponding elements of B, then A is thus made completely indistinguishable from B.

[Tacit premise] If two sets are indistinguishable, they have the same cardinal number.

--∴

∴ [Lemma 2] If there is a 1-1 correspondence between two sets A and B of changeable elements of the space-time world, it is “theoretically” possible to change the properties and mutual relations of the elements of A so that it has the same cardinal number as B.

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a tacit premise

Assume humanity survives forever; each individual dies, but there will be infinitely many generations.

www.hellhappens.com, from film "The Light of the World" by Jack Chick

Satan offers this choice:

(1) I will frequently and horribly torture everyone who is born on a Wednesday from this day on, or

(2) I will frequently and horribly torture everyone who is born on a Monday, Wednesday, or Friday, give YOU untold riches, and reveal to you the deepest secrets of the universe.

Prima facie it seems that (2) is worse because many *more* people are tortured

a moral thought experiment

BUT, there's a 1-1 correspondence between the Wednesday children and the Monday-Wednesday-Friday children.

So what if we dress up each Monday-Wednesday-Friday child to resemble a corresponding Wednesday child? Would that make (2) no worse than (1)?

What if we made them as alike as possible?

- Plastic surgery
- Brain configuration

So maybe sometimes haecceity matters

It's not *obvious* that indiscernibility always implies equal number, and Gödel gives no argument

would indistinguishability matter?

Gödel’s premises:

- not well known facts
- not widely acknowledged beliefs

Their appeal is *intuitive*

But Gödel ignores *other* strong intuitions, especially Part–Whole

- …which GREAT minds couldn’t shake
- Surely as analytic as the Hume–Cantor principle

His argument ignores the possibility of ***over***determination

- Part–Whole and Cantor’s Principle are *both* highly intuitive
- Intuitions can conflict
- So they’re not trustworthy

intuitions

So…

Gödel's argument fails to show that Cantorian power is the *uniquely correct* theory of set size

BUT, his tacit premise can be adapted to show that other theories are distinctly limited in *epistemic utility*

pragmatic considerations

A useful theory is an *informative* one

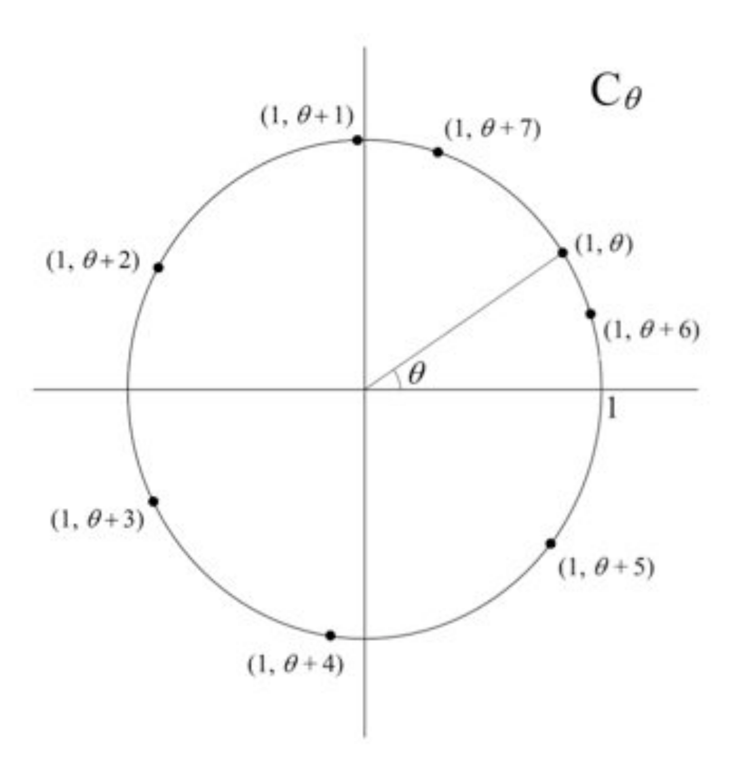
- Informative about facts or about consequences of other theories
- E.g., Cantor's powers tell us about 1-1 correspondence, and thereby, measure, probability, etc.
  - If two sets differ in power, this indicates a substantive difference that is independent of any notion of size

If sets that are exactly alike in their intrinsic properties and internal relations are *not* equal in size, then size doesn't mean much!

informativeness

- $C_\theta = \{(1, \theta), (1, \theta + 1), (1, \theta + 2), \ldots\}$
- $R(1, \sigma) = (1, \sigma + 1/2)$
- $RC_\theta = \{(1, \theta + 1/2), (1, \theta + 3/2), \ldots\}$
- $RRC_\theta = \{(1, \theta + 1), (1, \theta + 2), \ldots\} \subset C_\theta$

So on Euclidean theories, RRC_θ is *smaller* than C_θ



“Set Size and the Part-Whole Principle”, *Review of Symbolic Logic,* CUP

example

But RRC_θ is *just a rotation* of C_θ

- Elements are exactly alike in intrinsic properties and mutual relations
- So if a theory gives them different sizes, those sizes don't tell us much about the sets

RC_θ is *disjoint* from both C_θ and RRC_θ, but must be unequal in size to at least one of them

- Differing Euclidean sizes don't even indicate inclusion – they're largely *arbitrary*

informativeness

- Gödel's argument from *intuitions* to absolute *truth* fails

- But a parallel argument from *results* to *limitations of epistemic utility* succeeds

Euclidean set sizes are not necessarily *wrong*, but their usefulness is limited by arbitrariness and uninformativeness

conclusion

http://saipancakes.blogspot.co.uk/

thank you

gödel's argument for cantor’s cardinals

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