In search of $\aleph_0$- how infinity can be created

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Structure of the talk

1. A short history of the philosophy of $\aleph_0$
2. All mathematical infinities are potential
3. How the human mind creates numbers
4. How the human mind can create infinity
5. Developing the hypothesis

Mathematics in general epistemology

- The main problem of Platonism in epistemology: it is not how we in general get knowledge.
- To simplify a bit, the history of epistemology is the history of one great folly: rationalism.
- Mathematics has been the last discipline to resist the empiricist tempations because of its special character.
- In mathematics we seem to have essentially non-empirical ways of gaining knowledge.
- Some of this knowledge seems to be about the world, at the very least in the way it is used in empirical sciences.

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Breaking a tradition

- But with this tradition come two problems:

1. A methodological one: empirical studies of mathematical thinking are undervalued. There are practically no mentions, for example, of such studies in textbooks on philosophy of mathematics.
2. A rationalist bias: the rationalist epistemology of mathematics - even with its problematic position in general epistemology - is often taken as the default theory that should first be refuted.

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Aristotelean infinity

"For generally the infinite has this mode of existence: one thing is always being taken after another, and each thing that is taken is always finite, but always different.", Physics, Book 3: Chapter 6.

- Clearly the set of natural numbers - with the standard Dedekind-Peano axiom that if n is a number, then $n + 1$ is a number - fits Aristotle's characterization.
- But it is obvious that Aristotle is not describing an infinite *set*. Instead, he's describing a potentially infinite *process*.
- Thus Aristotle believed only in *potential* infinity and was followed by more than two millennia of tradition in which infinity in mathematics was considered to be potential.

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Cantor's paradise

- This was changed by Cantor who treated infinite sets as actual.
- Such was the break in tradition that he had to invent a new term for the numbers defined by infinite sets. Hence we still talk about *transfinite* ordinals and cardinals
- The lowest transfinite cardinal $\aleph_0$ is then the cardinality of the set of natural numbers.
- There is no denying the fruitfulness of Cantor's approach, especially after his proof that the set of reals is strictly larger than that of natural (or rational) numbers.

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Finitist mathematics

- Yet it is now easy to forget just how unintuitive the result must have seen. What does it mean that one infinity is strictly larger than another? For one unacquainted with higher mathematics, this may still seem absurd.
- Not everybody accepted this. One of the main tenets of Kronecker, Brouwer and their intuitionist followers was that there are no actual infinities (with the possible exception of $\aleph_0$).
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The ease of potential infinity

- For Cantor, the analysis on infinities led to God. Transfinite numbers were “forms or modifications of the actual infinite”.
- But how about the rest of us? We treat infinities as actual, but are we simply ignoring a valid philosophical question? We may talk about *completed* or *definite* infinity, but this does not hide the real issue.
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Whither infinity?

- If we use actual infinities in mathematics, in philosophy of mathematics we must strive to explain their ontological status.
- How should we, for example, treat the - according to modern science - perfectly valid possibility that the universe is finite?
- Mathematicians (and philosophers) are quick to note the difference between actual and *physical* infinity, but what does that difference mean if we do not believe in some form of Platonism about mathematical objects.
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All infinities are potential

- Let us start from an epistemological thesis: all mathematical infinities are fundamentally potential.
- The thesis is based on two things, the first of which cannot be fully argued for here:

1. Platonism is epistemologically problematic. If there are actual infinities somewhere in the world, how can we have epistemic access to them?

2. As I hope to show, we can outline a perfectly plausible epistemological account of mathematical infinities without presupposing actual infinities.

- The thesis is not a metaphysical one: it does not claim that actual infinities do not exist.

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- Rather, the thesis is purely epistemological: even if actual infinities exist, that is not where we get the concept of infinite into mathematics.
- Even in the case of there existing physical infinity, it should be reasonable to assume that we don't have epistemic access to the universe as a totality.
- Thus all our mathematical considerations about infinity are based on reasoning about finite collections. It is by first considering finite sets of natural numbers that we eventually establish that there are potentially infinitely many of them.
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The logicist tradition

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- At least ever since Frege's criticism of psychologism in his *Grundlagen*, the standard view in philosophy has been one in which the epistemology of natural numbers is essentially (if not heuristically) removed from empirical aspects.
- The idea for Frege was to derive laws of arithmetic from logic, i.e., laws of thought. This way, there would be no need for special numerical cognition.

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A naturalist turn

- As is well known, Frege's Basic Law V is inconsistent. The best effort to revive Frege's programme, neo-logicism of Wright & Hale, has to use a non-logical law to define equinumerosity (so-called Hume's Principle).
- However, even if Frege's logicist programme would have been successful, it now seems that he would have been too strict about what laws of thought are.
- Recent empirical research suggests strongly that human beings have forms of numerical cognition that should not be reduced to logical laws like the ones in Frege's system.
- This cognition can already be seen in small infants and we share it with many animals. (Cf. Dehaene 2011 and Dehaene & Brennan (eds.) (2011).

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Empirical data

- In the often replicated Wynn experiment, infants are surprised when they see a setting corresponding to the unnatural arithmetic $1 + 1 = 1$.
- Goldfish can learn to recognize the number of objects, even when the combined surface area, total illumination, etc. of the objects is the same.
- Rats can learn to recognize the number of tones, distinguishing it from the total duration of the tones.
- These and numerous other experiments suggest that human infants and many nonhuman animals not only have the capability but also a natural tendency to use numerosities in order to categorize observations.

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Approximate Number System

- This primitive, proto-arithmetical ability is often called the Approximate Number System (ANS).
- ANS is thought to consist of two parts:

1. The ability to *subitize*, to recognize quantities without counting.

2. An "analog magnitude system," a way of keeping track of quantities in the working memory.

- Both of these parts quickly lose accuracy as numbers get bigger. Usually this happens already at 4-5 in the case of subitizing and before 10 for the analog magnitude system.

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ANS as basis for arithmetic

- So to call ANS arithmetic - as many scientists starting from Wynn have - is grossly misleading.
- At the same time, ignoring ANS as a relevant proto-arithmetical system for operating quantities should no longer be acceptable.
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- According to this hypothesis, central to the move from proto-arithmetic to arithmetic is the realization that just like 1, 2, 3, 4 and 5 are unique successors of the previous numerosity in the sequence, this process can be continued: to *every* numerosity there is such a successor.

Infinity

- From the above account, it would be easy to deduce that the sequence of natural numbers must be infinite.
- But this would be too quick. Not every culture that has mastered large numbers (e.g. the Mayans) included the notion of infinity in their arithmetic.
- There seems to be nothing inevitable in the concept of infinity of numbers. Being able to generalize on the concept of successor does not automatically evoke infinity.
- On the other hand, there are various non-mathematical (religious etc.) ways of forming a concept of infinity.
- Having informal understanding that natural numbers have successors does not seem to be either necessary or sufficient condition for the position that there are infinite things.

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- But this would be too quick. Not every culture that has mastered large numbers (e.g. the Mayans) included the notion of infinity in their arithmetic.
- There seems to be nothing inevitable in the concept of infinity of numbers. Being able to generalize on the concept of successor does not automatically evoke infinity.
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- They first reject the negative explanations ("not-finite"):

"...this does not give us any of the richness of our conceptions of infinity. Most important, it does not characterize infinite things: infinite sets, infinite unions, points at infinity, transfinite numbers. To do this, we need not just a negative notion ("not finite") but a positive notion - a notion of infinity as an entity-in-itself." (p. 155, Italics in the original)

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- Since the negative definition would seem to suffice for potential infinities, Lakoff & Núñez clearly want to explain the mathematical notion of *actual* infinity.
- Their hypothesis is that all cases of actual infinity are applications of one conceptual metaphor, the "Basic Metaphor of Infinity" (BMI).
- The general idea of BMI is that processes that can be continued indefinitely are thought to have an ultimate result.
- Just like we speak of the final resultant state of a completed iterative process (such as counting to ten), with the help of BMI we can speak of the metaphorical "final resultant state" of iterative processes that go on and on.

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Trouble with BMI

- But is the metaphor stretched too much here? We can reach the number ten by counting from one without any deeper understanding of the process. But we can only understand the "final resultant state" of an indefinite counting process by understanding the infinity of natural numbers.
- We are talking about two different kinds of processes here. One where the process is counting, the other where the process is understanding the *nature* of counting.
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Processes as objects

- The difference between applying a process and understanding a process is crucial to the matter at hand: in the latter case processes are treated mathematically as *objects*.

Case study: the Fibonacci sequence

- Take the Fibonacci sequence as an example: $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$
- The lesson from Kripkenstein: no finite part of the sequence will give us the rule which the sequence follows.
- But the recursive definition of the Fibonacci sequence $F(0) = 0, F(1) = 1$ and for all n: $F(n) = F(n-1) + F(n-2)$ clearly does. It defines the Fibonacci sequence as an *object* in mathematics.

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Another metaphor

- Rather than BMI, I claim that this is the metaphor we use. We take a *process* of forming numbers in the Fibonacci sequence and we talk of the *object* defined by that process - in this case, an infinite set.
- Clearly it is a metaphor: infinite processes are not objects and we cannot talk of the end product of infinite processes. Yet we talk of objects defined by potentially infinite processes.
- But this "process $\rightarrow$ object" -metaphor is fundamentally nothing more than Aristotle's idea of potential infinity. It only uses the understanding that the Fibonacci sequence, defined recursively as above, is endless, i.e. *not finite*.
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How $\aleph_0$ is created minimally

- These three things would seem to be enough for us to reach $\aleph_0$, the cardinality of the lowest actual infinity.

1. We need to have some conception of a process being endless.

2. We need to have some conception of axiomatization, recursion, or implicit definition in order to express endless processes with no ambiguity.

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Infinity in set theory

- In set theory, there is of course an *axiom* of infinity to give us the existence of the set of natural numbers (the inductive set): $\exists S(\emptyset \in S \wedge \forall x \in S((x \cup \{x\}) \in S)$.
- The axiom of infinity clearly uses the "process $\rightarrow$ object" metaphor. With the axiom, we move from the process to the existence of the infinite set.
- This way, the present explanation of the creation of infinity is both epistemologically and ontologically minimal, but it also corresponds to set theoretic practice.

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The cognition of infinity

- Lakoff & Núñez make it clear that theirs is a hypothesis without direct empirical support. While there is empirical support for ANS being foundation for arithmetic, the step to infinity is cognitively too complicated to study at the moment.
- Consequently, we can at this point only assess the philosophical plausibility of each hypothesis.
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- Once we establish that Cantor's actual infinities can be given a satisfactory metaphorical interpretation, we can continue with his analysis of transfinites without any problems.
- Under such a theory, the transfinite cardinal $\aleph_1$- according to the Continuum Hypothesis - the cardinality of the set of real numbers, comes by continuing the metaphor of processes as sets.

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Conclusions

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- Such an account is open to various epistemological interpretations. The main focus has been on showing that we can have an epistemologically plausible account of infinity without assuming that we have epistemic access to infinite things. This does not refute the possibility of such access.
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