Continuous vs. Discrete Infinity in Foundations of Mathematics and Physics

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Outline

Two Conceptions of Infinity

- There are two different conceptions of infinity in foundations of mathematics and physics.
- One is set-theoretical or Cantorian, and regards an infinity (especially, continuum) as an enormous amount of discrete points or elements.
  - “Discrete" means those points exist independently of each other, and there is no cohesiveness among them. Space continuums consist of massive numbers of discrete points.
- The other is geometric or Brouwerian, and considers an infinity like a continuum to be a cohesive totality, or rather a finitary law to generate it (in infinite time).
  - This gives rise to intrinsic continuity as seen in Brouwer’s theory of choice sequences. Space continuums are cohesive totalities in the limits of generating processes.

Category Theory as Geometry

Category-theoretical foundations of mathematics support the geometric view on infinity.

- Indeed, topos theory gives categorical models of Brouwer's intuitionistic mathematics, in particular his continuity principle.
- Homotopy type theory yields fibrational models of Martin-Loef's intuitionistic type theory with its identity type intensional rather than extensional.

The distinction between the Cantorian and Brouwerian conceptions of infinity would be more or less parallel to that between Aristotle's ideas of actual and potential infinity.

The Aim of the Talk

Here I aim at the following:

- Elucidating conceptual underpinnings of the dichotomy between

  Cantorian extensional discrete infinity

  and

  Brouwerian intentional continuous infinity

  by placing it in a wider context of (both analytic and continental) philosophy.
- In particular, shedding new light on the concept of space continuums among different ideas of infinity.

There is categorical duality b/w the two conceptions of infinity (categorical duality theory is my main field).

Different Ideas on Space

Since the ancient Greek philosophy, there have been a vast number of debates on whether or not the concept of points precedes the concept of the space continuum.

- On the one hand, one may conceive of points as primary entities, and of the continuum as secondary ones to be understood as the collection of points.
- On the other, the whole space continuum may come first, and then the concept of a point is derived as a cut of it.
- We basically have two conceptions of space: the point-set and point-free ones.

This is more or less analogous to the well-known dichotomy b/w Newton's absolute space and Leibniz's relational space.

Outline

Wittgenstein's View on Space

Wittgenstein gives a fresh look at the issue of the relationships between space and points:

> *What makes it apparent that space is not a collection of points, but the realization of a law? (*Philosophical Remarks*, p.216)*

Wittgenstein's intensional view on space is a compelling consequence of his persistent disagreement with the set-theoretical extensional view of mathematics:

> *Mathematics is ridden through and through with the pernicious idioms of set theory. One example of this is the way people speak of a line as composed of points. A line is a law and isn't composed of anything at all. (*Philosophical Grammar*, p.211)*

What is a Law?

First of all, what Wittgenstein calls a law should be clarified.

- “In order to represent space we need – so it appears to me – something like an expansible sign" (PR, p.216)
- What precisely is a sign, then?
- He proceeds in the same page: it is “a sign that makes allowance for an interpolation, similar to the decimal system."
- He then adds, “The sign must have the multiplicity and properties of space."

E.g., think of expanding digital sequences, such as:

0.1 $\rightarrow$ 0.11 $\rightarrow$ 0.110 $\rightarrow$ 0.1101 $\rightarrow$ ...

Coin-Tossing Game

To elucidate what he means, the following discussion on a coin-tossing game seems crucial:

> *Imagine we are throwing a two-sided die, such as a coin. I now want to determine a point of the interval AB by continually tossing the coin, and always bisecting the side prescribed by the throw: say: heads means I bisect the right-hand interval, tails the left-hand one. (*PR*, pp.218-219)*

It is crucial that a point is being derived from the coin-tossing game, a sort of law, which Wittgenstein thinks realises space. A point is merely a secondary entity. The law to determine a point in its limiting process is the primary one.

Processes = Points

The process of tossing the coin, of course, does not terminate within finite time. It may be problematic, since Wittgenstein takes the position of ultrafinitism.

- So, Wittgenstein remarks, “I have an unlimited process, whose results as such don’t lead me to the goal, but whose unlimited possibility is itself the goal" (PR, p.219).
- To put it differently, such a rule for determining a point only gives us the point in infinite time, but still we may regard a rule itself as a sort of point.
- This idea of identifying points with rules or functions is now standard in mainstream mathematics, such as Algebraic and Non-Commutative Geometry.

Note: a shift of emphasis is lurking behind the scene, from static entities like points to dynamic processes like laws.

The Cantor Space $\mathbf{2}^\omega$

- If you are familiar with Brouwer's theory of the continuum, you would notice there is a close connection between Brouwer's and Wittgenstein's views on space.
- Wittgenstein's coin-tossing game almost defines the Cantor space $\mathbf{2}^\omega$ in terms of contemporary mathematics (where $\mathbf{2} = \{0, 1\}$).
- The Cantor space $\mathbf{2}^\omega$ is the space of infinite sequences consisting of zeros and ones, which in turn correspond to heads and tails of a coin in Wittgenstein's terms; actually, he himself discusses this correspondence (PR., p.220).

Brouwer's Concept of Spread

Now let me quote a passage by Brouwer which, together with the quotations above, exhibits a remarkable link between Brouwer's and Wittgenstein's ideas of space (Brouwer 1918, p.1; translation by van Atten 2007):

> *A spread is a law on the basis of which, if again and again an arbitrary complex of digits [a natural number] of the sequence ζ [the natural number sequence] is chosen, each of these choices either generates a definite symbol [..] Every sequence of symbols generated from the spread in this manner (which therefore is generally not representable in finished form) is called an element of the spread.*

A spread is basically a law to generate a sequence of symbols.

Comparing Brouwer's Space with Wittgenstein's

- For Brouwer, a law is a rule to make a sequence of countably many digits. All such sequences together yield the so-called Baire space ω^ω.
- The difference between Brouwer's and Wittgenstein's laws only lies in which to use two digits only (or $\mathbf{2}^\omega$ in modern terms) or all natural numbers (or ω^ω).
- Although this in fact gives rise to a certain technical difference, however, the reals $\mathbb{R}$ ($[0, 1]$) can be expressed in the same way in both cases, and there is no doubt that the underlying conceptual view of capturing the concept of space in terms of laws is fundamentally the same.

Ramifications of the Point-Free Space

- It may thus be concluded that Wittgenstein's and Brouwer's conceptions of space build upon the same core idea of regarding space as a law to form infinite digital sequences.
  - Still there are important differences in the light of Wittgenstein's distinction between arithmetical and geometrical space.
- Their philosophically motivated idea has now become a standard method, in Computer Science, to implement exact computation over continuous infinitary structures.
  - Brouwer's Continuity Principle states: every function from $\mathbb{R}$ to $\mathbb{R}$ is continuous. It is strange for non-intuitionists.
  - This has a plausible computational interpretation: every computable function from $\mathbb{R}$ to $\mathbb{R}$ is continuous. This perfectly makes sense for classical mathematicians as well.
- Formal Topology would be the most recent development.

Brouwer's Influence on Wittgenstein

Brouwer's influence on Wittgenstein is well known in general.

- Rodych (2011): "There is little doubt that Wittgenstein was invigorated by L.E.J. Brouwer's March 10, 1928 Vienna lecture."
  - Some say he did not attend it, but later read the script.
- Wittgenstein does not explicitly mention Brouwer in his discussion on space.
- Taking Wittgenstein's illustration into account, however, I think Wittgenstein's conception of space in particular may have been influenced by Brouwer's continuum theory.
- This may thus be yet another case of Brouwer's influence on Wittgenstein wrt. the concept of space in particular.

But I am not a historian. I would like to hear experts' opinions.

Outline

Algebra = Space

The shift from point-set to point-free sp. is not just a phil. issue.

- It was crucial in Modernisation of Mathematics.
  - Disclaimer: no direct influence from WIttgenstein or Brouwer. Still, the underlying idea is much the same.
- Topological spaces (manifolds, varieties, etc.) are based upon the view of space as collections of points.
- The algebras of functions on topo. spaces turn out to keep the same amount of information as the original point-set spaces (under certain conditions).
  - We can recover the points from the algebraic structures.
  - Intuitively: algebra = structure of regions; then, points = limits of shrinking regions (= prime ideals of algebras).
- Algebra itself has finally become considered to be space.

It is Algebraisation of Geometry. It is especially indispensable for Non-Commutative Geometry and Quantum Mechanics.

Is “Modern" Better than “Premodern"?

- The 20th century’s modern “conceptual" math discarded quite some of the 19th century’s “computational" math.
- Schönberg says, “There is still plenty of good music to be written in C major." The same remark applies to mathematics (J. Gray, *Plato’s Ghost*, p.39).
- I am not saying the shift is that from “bad" to “good". Indeed, some revival of 19th century math has occurred after the dominance of “abstract non-sense" math.

Categorical Duality

Mathematically we have a category-theoretical duality b/w set-theoretical, point-set and algebraic, point-free conceptions of space. Diverse categorical dualities (i.e., dual categ. equiv.):

	Ontic	Epistemic	Duality
Logic	Semantics	Syntax	Stone
Topology	Points	Opens (Prop.)	Isbell, Papert
Alg. Geometry	Variety	Ring	Hilbert, Gro.
Computer Sci.	Denotations	Behaviour	Abramsky
Quantum Phys.	State	Observable	Gelfand et al.

Dualities in diverse fields have “something" in common. I think duality arises between the ontic and the epistemic. Duality tells us point-set and point-free geometries are essentially equiv.

Origin‘s’ of Duality

Similar ideas by a toposopher and a (genuine) philosopher:

- Lawvere: “duality b/w the conceptual and the formal". He founded Topos Theory, together with Grothendieck, Tierney.
  - space-algebra duality, semantics-syntax duality, and state-observable duality would fall into this picture.
- Granger: “duality b/w objects and operations."
  - In Quantum Mechanics: duality b/w Dirac’s “bra" $\langle\varphi|$ and “ket" $|\varphi\rangle$; this is duality b/w what acts and what is acted on, or duality b/w subjects and objects.

Modernisation, I think, is the shift from “objects" to “operations". Where is the origin of such shifts? May be no single origin.

- One origin could be Cassirer’s “*Substance and Function*". Whitehead’s process philosophy was later than Cassirer’s.

Even Gödel thought of the shift from “substance" to “function."

Gödel's shift from the right to the left

Gödel's "The modern development of the foundations of mathematics in the light of philosophy" in his *Collected Works*:

> *the development of philosophy since the Renaissance has by and large gone from right to left ... Particularly in physics, this development has reached a peak in our own time, in that, to a large extent, the possibility of knowledge of the objectivisable states of affairs is denied, and it is asserted that we must be content to predict results of observations. This is really the end of all theoretical science in the usual sense ...*

In the (quantum) physical context, Gödel's "right" seems to mean reality or substance, and "left" observational phenomena. In other contexts, Gödel says Metaphysics is "right"; Logic is "left." The right $\sim$ the ontic; the left $\sim$ the epistemic.

Concluding Remarks

In the light of duality:

- Two conceptions of infinity: Cantorian extensional discrete infinity and Brouwerian intentional continuous infinity.
- Categorical duality b/w them. Rich instances: semantics-syntax duality, (quant) state-observable duality, variety-ring duality, duality b/w topo. sp. and formal sp., ...
- Duality b/w the ontic and the epistemic gives a unifying perspective on diverse categorial dualities. General theories of such dualities exist as well.

Concluding Remarks (cont.)

In the light of the conceptual shift:

- Wittgenstein's phil. of space may be understood in terms of the shift from point-set to point-free sp. (points to laws).
- It seems to be part of the larger, trans-disciplinary, conceptual shift, from “right" to “left", from substance to function, or from static objects to dynamic processes.
- Such shifts have played crucial rôles in Modernisation of Math and Physics. Even the Linguistic Turn in Philosophy (reality to language) could count as a case of the shift.
    - Wittgenstein: “words are not a translation of something else that was there before they were" (*Zettel*, p.33).
- The shift from the ontic to the epistemic might be a distinctive characteristic of Modernisation in general.