Reading About Cardinality

Matthew Inglis
Mathematics Education Centre
Loughborough University

Mathematics Education Centre MEC

Loughborough University

Plan

- Students find cardinal numbers difficult, but why?
- Existing explanations: theory of conceptual change.
- How do ‘strong’ and ‘weak’ students learn about cardinal arithmetic from text?
- Can mathematical reading strategies be improved?

Cardinal Numbers

- Cardinal numbers are known to be hard for students to understand (e.g., Borasi, 1984; Duval, 1983; Fischbein, Tirosh & Hess, 1979; Fischbein, Tirosh & Melamed, 1981; Tall, 1980, 1981, 2001a, 2001b; Tirosh, 1985).
- Traditionally understood from a “conceptual change” perspective.
- Goal of the talk: to argue that this should be supplemented by a “many students don’t (but can) read effectively” perspective.

Conceptual Change

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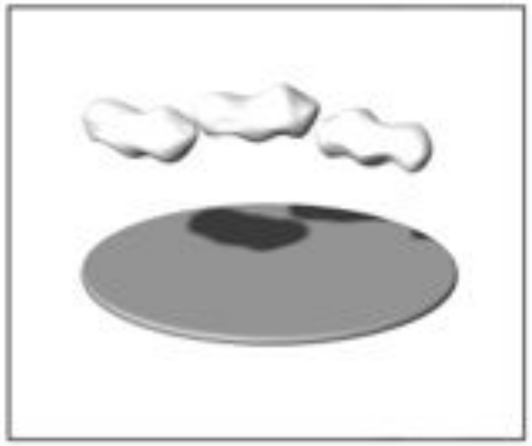
- When new knowledge comes into conflict with existing knowledge major reorganisation is required, referred to as conceptual change.
- Paradigmatic example: children learning about the shape of the earth.

Conceptual Change



Straatemeier et al. (2008)

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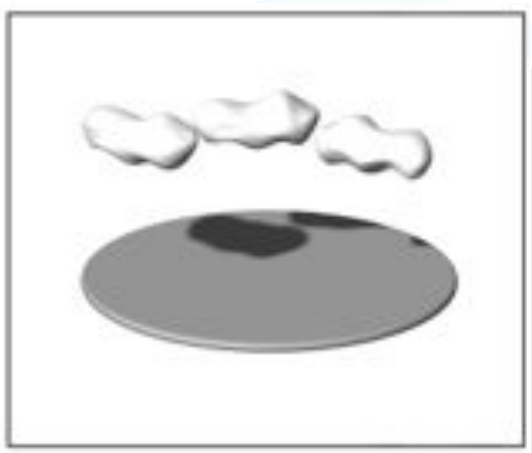

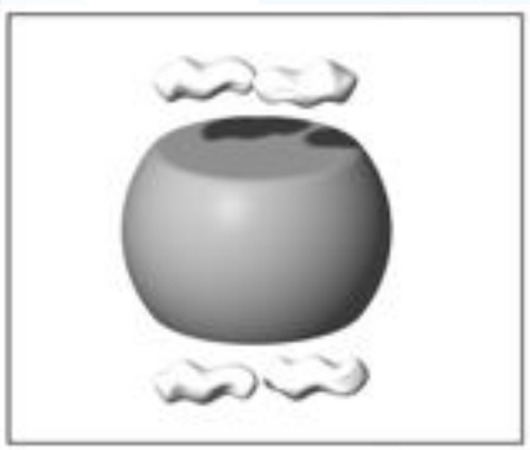

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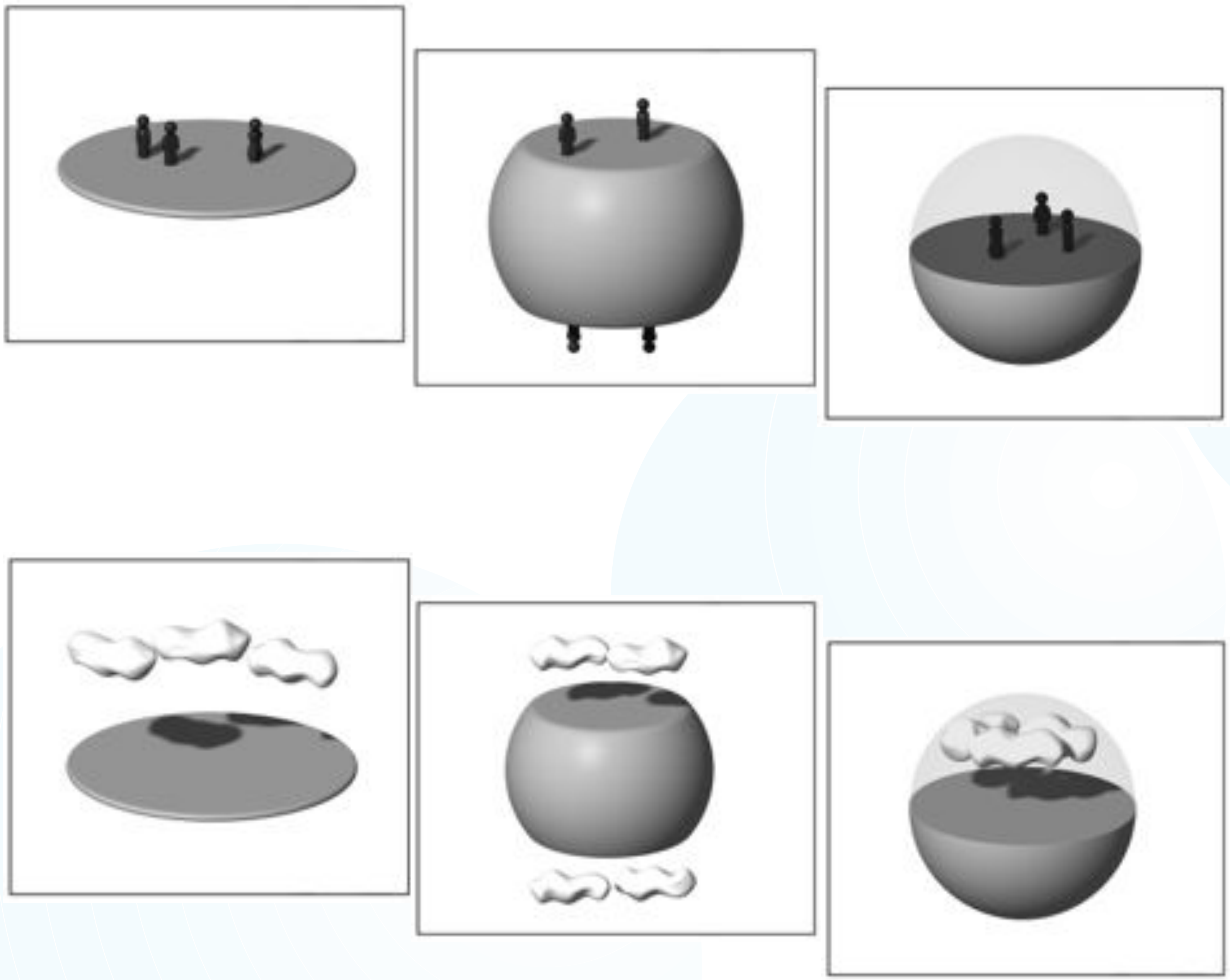

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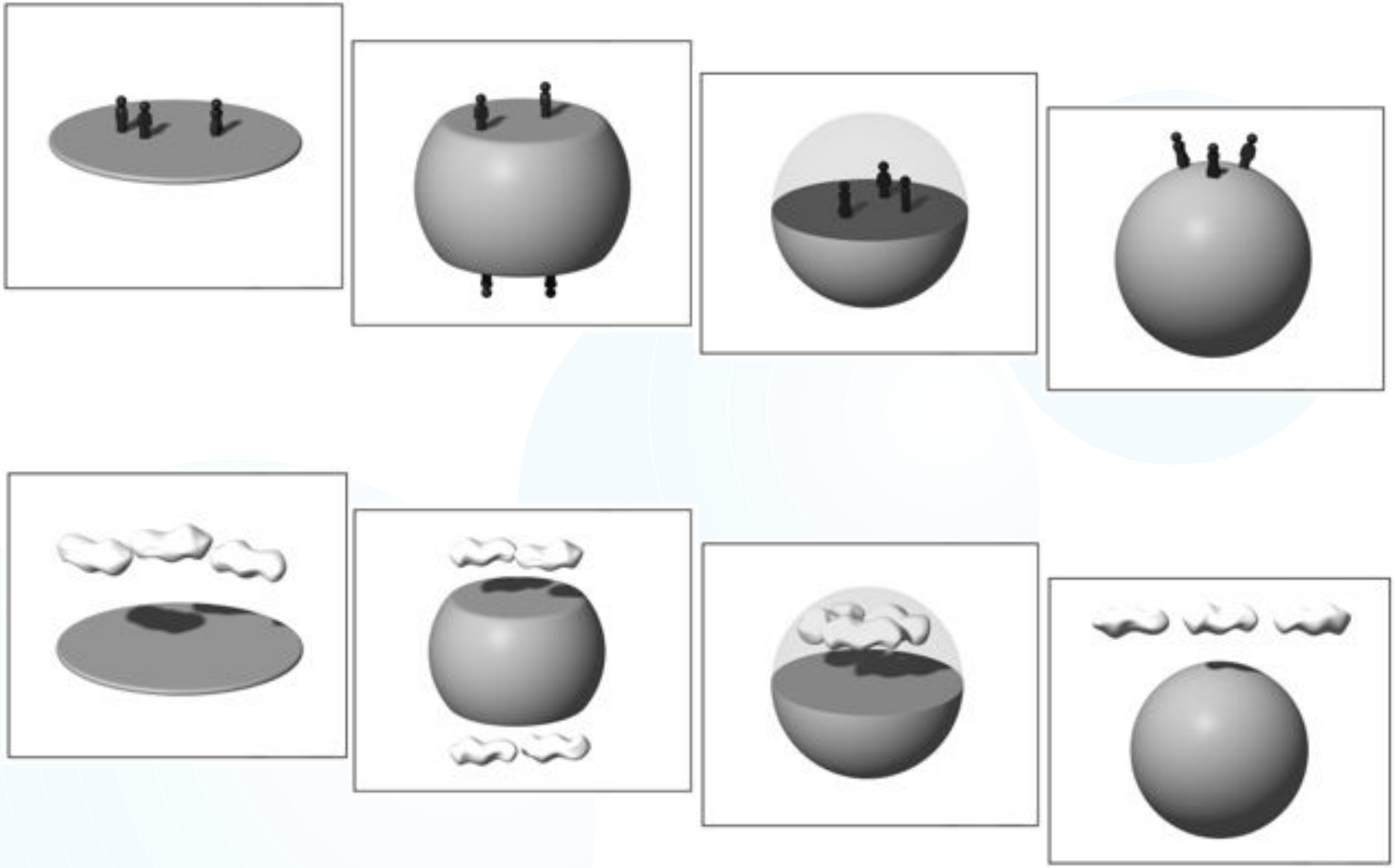

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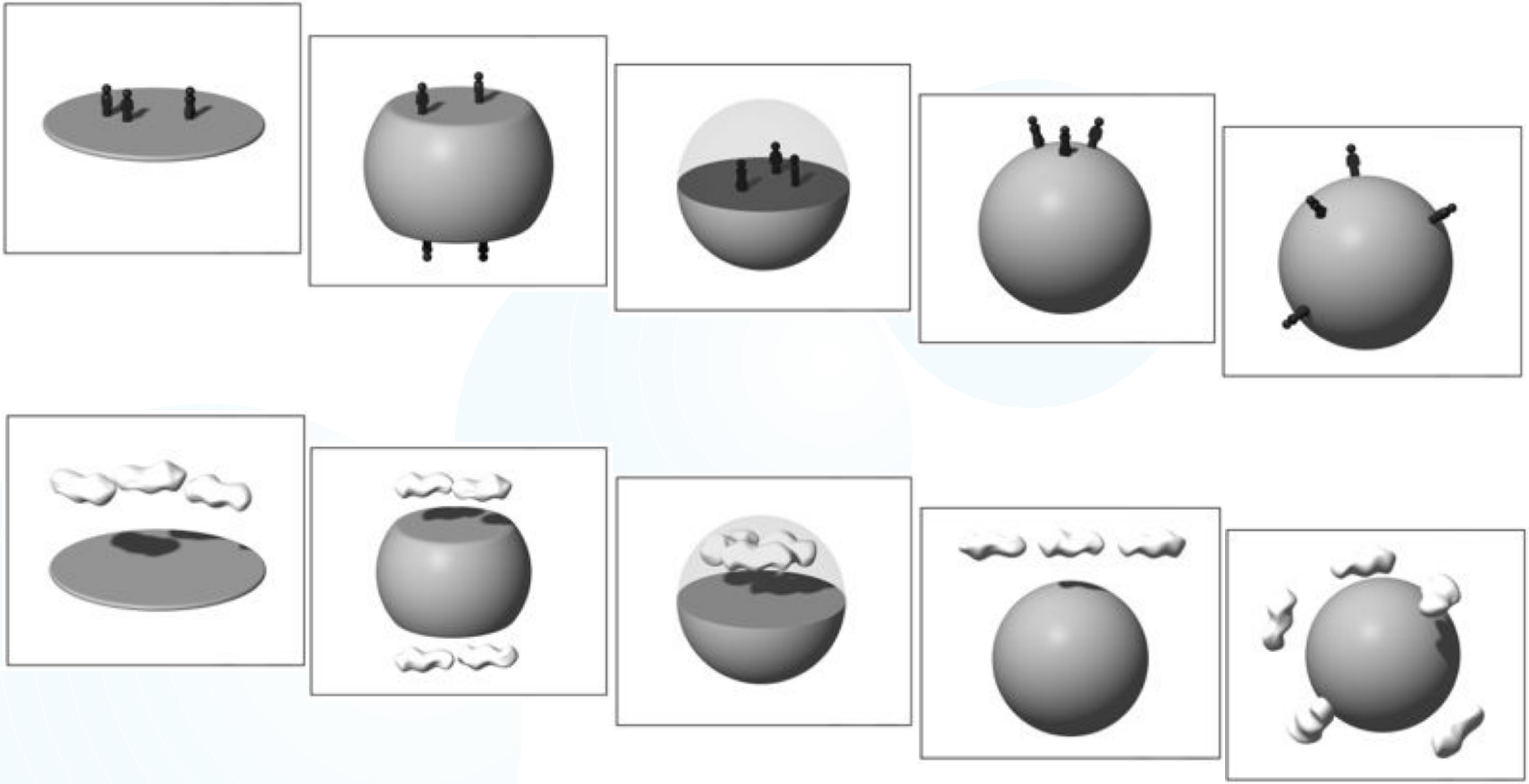

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Conceptual Change and Cardinality

Major differences between students' existing understanding of how numbers behave and how cardinal numbers behave.

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What is this? Isn't subtraction undefined?

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- Basic idea of the conceptual change approach is that you have to reconstruct your existing knowledge of numbers to account for these strange new properties.
- This is very difficult.
- But... university-level students are really experienced at changing their conceptualisation of number.
- For example: the “Kuhnian revolution” from naturals to rationals.

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We all have this bias, but (mostly) have successfully learnt to inhibit it

Natural Number Bias

Learning and Instruction 28 (2013) 64–72

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The natural number bias and magnitude representation in fraction comparison by expert mathematicians

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[b] *Heinz Nixdorf-Stiftungslehrstuhl für Didaktik der Mathematik, TUM School of Education, Technische Universität München, Marsstr. 20–22, 80335 München, Germany*[1]

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ABSTRACT

When school students compare the numerical values of fractions, they have frequently been found to be biased by the natural numbers involved (e.g., to believe that 1/4 > 1/3 because 4 > 3), thereby considering fractions componentially as two natural numbers rather than holistically as one number. Adult studies have suggested that intuitive processes could be the source of this bias, but also that adults are able to activate holistic rather than componential mental representations of fractions under some circumstances. We studied expert mathematicians on various types of fraction comparison problems to gain further evidence for the intuitive character of the bias, and to test how the mental representations depend on the type of comparison problems. We found that experts still show a tendency to be biased by natural numbers and do not activate holistic representations when fraction pairs have common numerators or denominators. With fraction pairs without common components, we found no natural number bias, and holistic representations were more likely. We discuss both findings in relation to each other, and point out implications for mathematics education.

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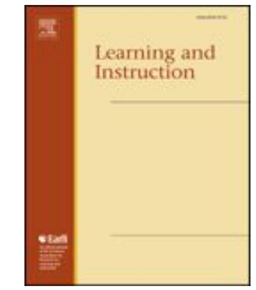

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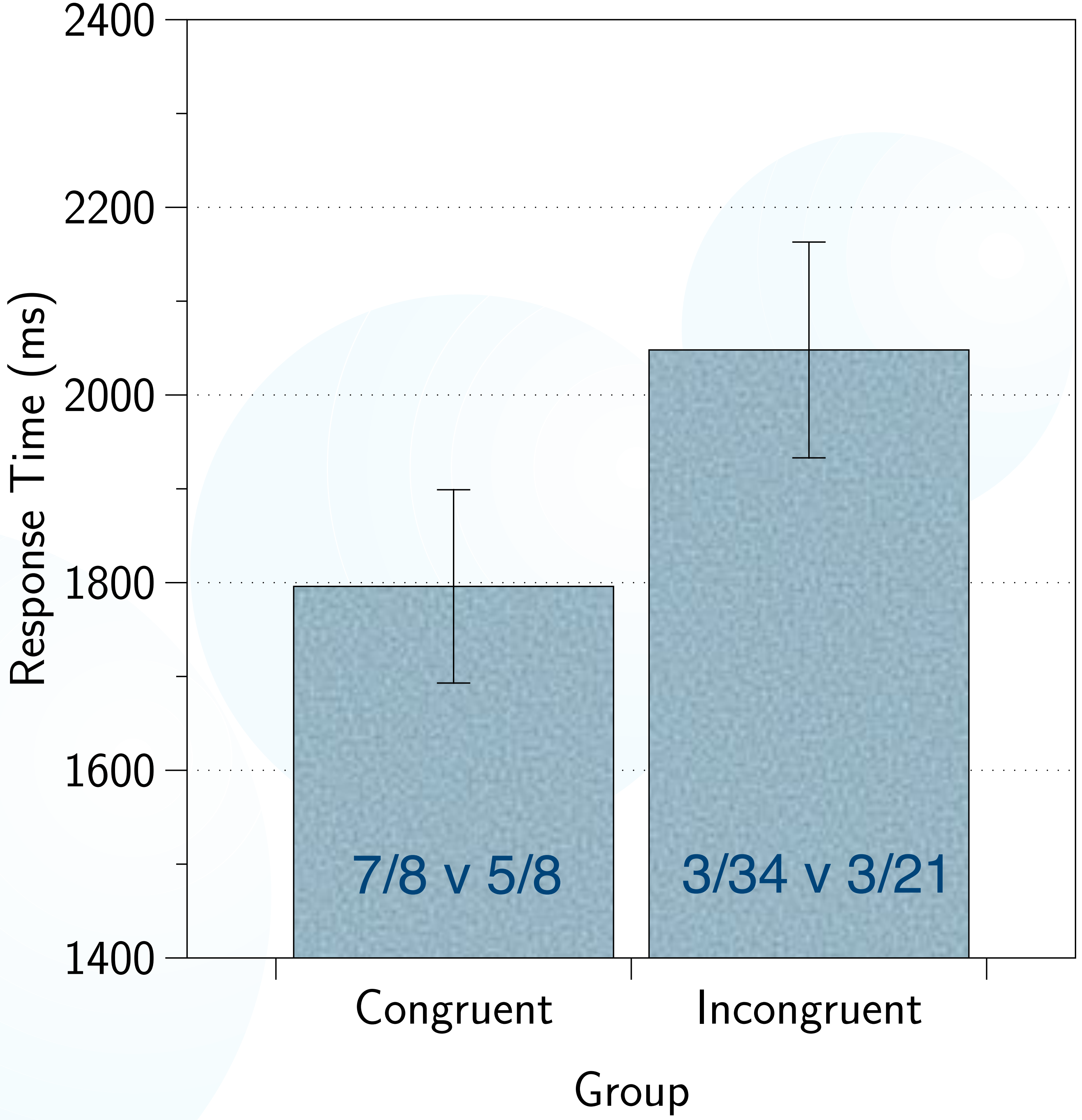

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- Students who meet cardinality have successfully negotiated the shift from naturals to integers, from integers to rationals, from rationals to reals, and from reals to complex numbers.
- They seem to do this by successfully inhibiting (not replacing) prior knowledge structures.
- It seems like another change in the meaning of “number” shouldn’t be too surprising to them?

Study 1: Rationale

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- What would confuse them?
- Would they get hung up with aspects of cardinal arithmetic that conflict with their prior knowledge of arithmetic?

Study 1: Method

- Recruited 20 students: 12 ‘good’ students (scored over 70% in their first year set theory module), and 8 ‘weak’ students (scored less than 60%);
- All had taken a set theory module which introduced cardinal numbers (but not cardinal arithmetic);
- Asked them to read an introduction to cardinal arithmetic (taken from Stewart & Tall, 1979);
- Recorded their eye-movements as they read;
- Then asked them to complete a comprehension test.

Basic Definitions

Definitions

Bijection. A bijection, or one-to-one correspondence, is a function $f : A \to B$ which is both injective and surjective. In other words f is a bijection if (i) for every $a_1, a_2 \in A$, $f(a_1) = f(a_2) \implies a_1 = a_2$ (injective); and (ii) if $f(A) = B$, in other words if for every $b \in B$ there exists an $a \in A$ such that $f(a) = b$ (surjective).

Cartesian Product. Let X and Y be sets. Then the Cartesian Product of X and Y, denoted $X \times Y$ is the set

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

So, for example, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$.

Cardinal Numbers

'What is infinity?'. When some first-year university students were asked this question recently, the consensus was 'something bigger than any natural number'. In a precise

Cardinal Arithmetic

Just as we can add, multiply and take powers of finite cardinal numbers, we can mimic the set-theoretic procedures involved and define corresponding operations on infinite cardinals. Some, but not all, of the properties of ordinary arithmetic carry over to cardinal numbers, and it is most instructive to see which ones. First of all the definitions:

Addition. Given two cardinal numbers α, λ (finite or infinite), select disjoint sets A, B such that $|A| = \alpha$, $|B| = \lambda$. Define $\alpha + \lambda$ to be the cardinal number of $A \cup B$.

Defined how to add, multiply and take powers of cardinal numbers

Multiplication. If $\alpha = |A|$, $\lambda = |B|$, then $\alpha\lambda = |A \times B|$.

Powers. If $\alpha = |A|$, $\lambda = |B|$, then $\alpha^\lambda = |A^B|$ where A^B is the set of all functions from B to A.

The reader should pause briefly and check that when the sets concerned are finite then this corresponds to the usual arithmetic. In particular, when $|A| = m$ and $|B| = n$, then on defining a function $f : B \to A$, each element $b \in B$ has m possible choices of

Now let's make some explicit calculations with cardinals. Because a countable union of countable sets is countable, we find that

$$n + \aleph_0 = \aleph_0 + n = \aleph_0, \text{ for any finite cardinal } n,$$

$$\text{and } \aleph_0 + \aleph_0 = \aleph_0.$$

Example Calculations

This shows us that we have no possibility of defining subtraction of cardinals where infinite cardinals are involved, for what would $\aleph_0 - \aleph_0$ be? According to the above results it could be any finite cardinal or $\aleph_0$ itself, so subtraction cannot be defined uniquely so that

$$\aleph_0 - \aleph_0 = \alpha \iff \aleph_0 = \aleph_0 + \alpha$$

Because the cartesian product of two countable sets is countable, it is easy to deduce that

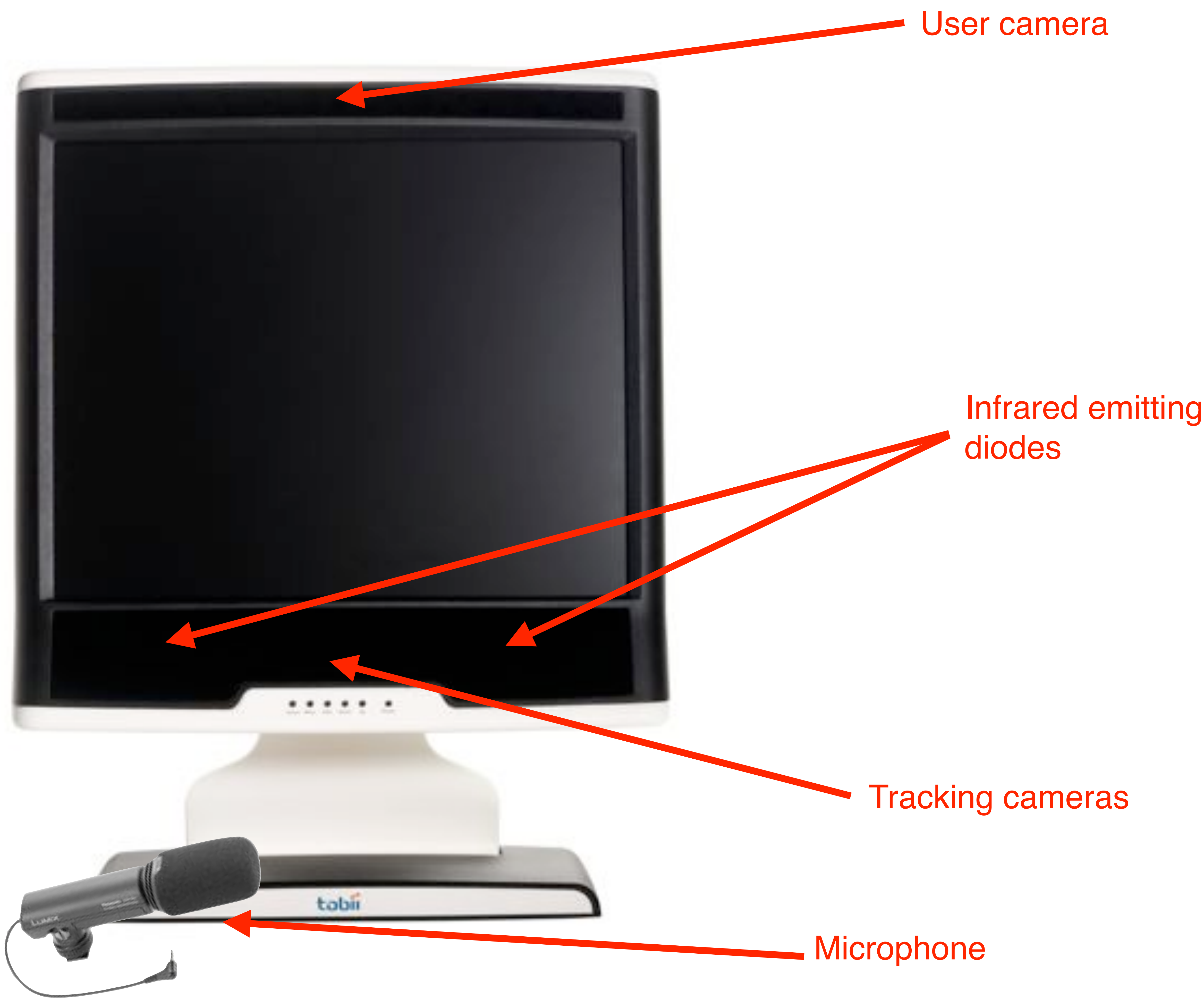
$$n\aleph_0 = \aleph_0 n = \aleph_0 \text{ for } n \in \mathbb{N}$$

$$\text{and } \aleph_0\aleph_0 = \aleph_0.$$

It is interesting to calculate $0\aleph_0$. This turns out to be zero. In fact we have

$$0\lambda = 0$$

for each cardinal number λ. This is because $A = \emptyset \implies A \times B = \emptyset$ for any other set B, for if A has no elements then there are no ordered pairs (a, b) for $a \in A$ and $b \in B$. This means that, in terms of cardinal numbers, zero times infinity is zero, no matter how big the infinite cardinal is.


User camera
Infrared emitting diodes
Tracking cameras
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- Just & Carpenter’s (1980) **eye-mind hypothesis**: there is a close correlation between eye position and visual attention position.

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- Number of Fixations: more fixations indicates less optimal search (Goldberg & Kotval, 1999).
- Large number of saccades between A and B: may indicate an attempt to find a connection between A and B (Inglis & Alcock, 2012).

Study 1: Analysis

Questions

1. Did the two groups learn different amounts?
2. Were there between-group differences in aspects of the text which conflicted with existing knowledge?
3. What else differed between the groups?

Preliminaries

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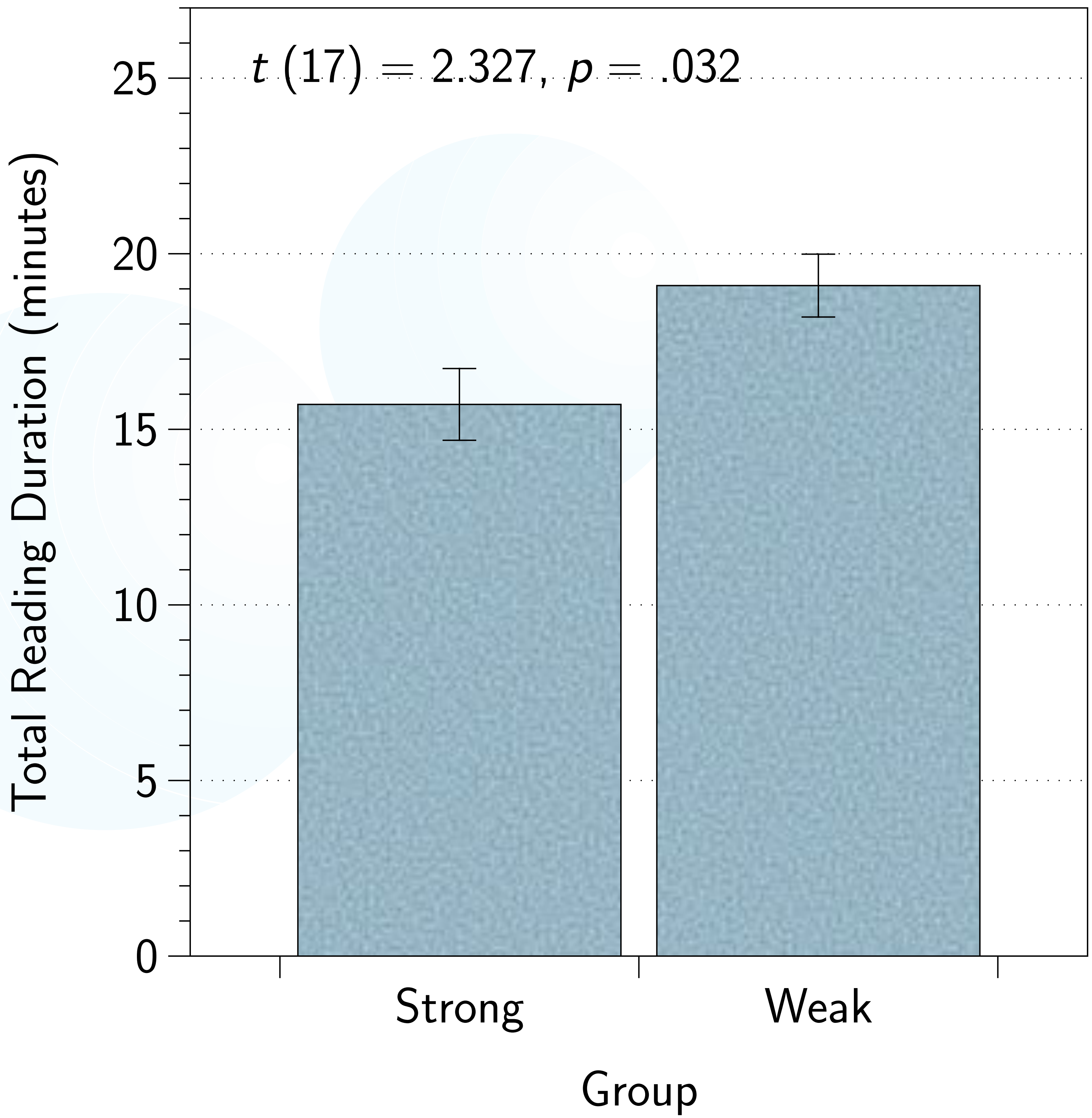

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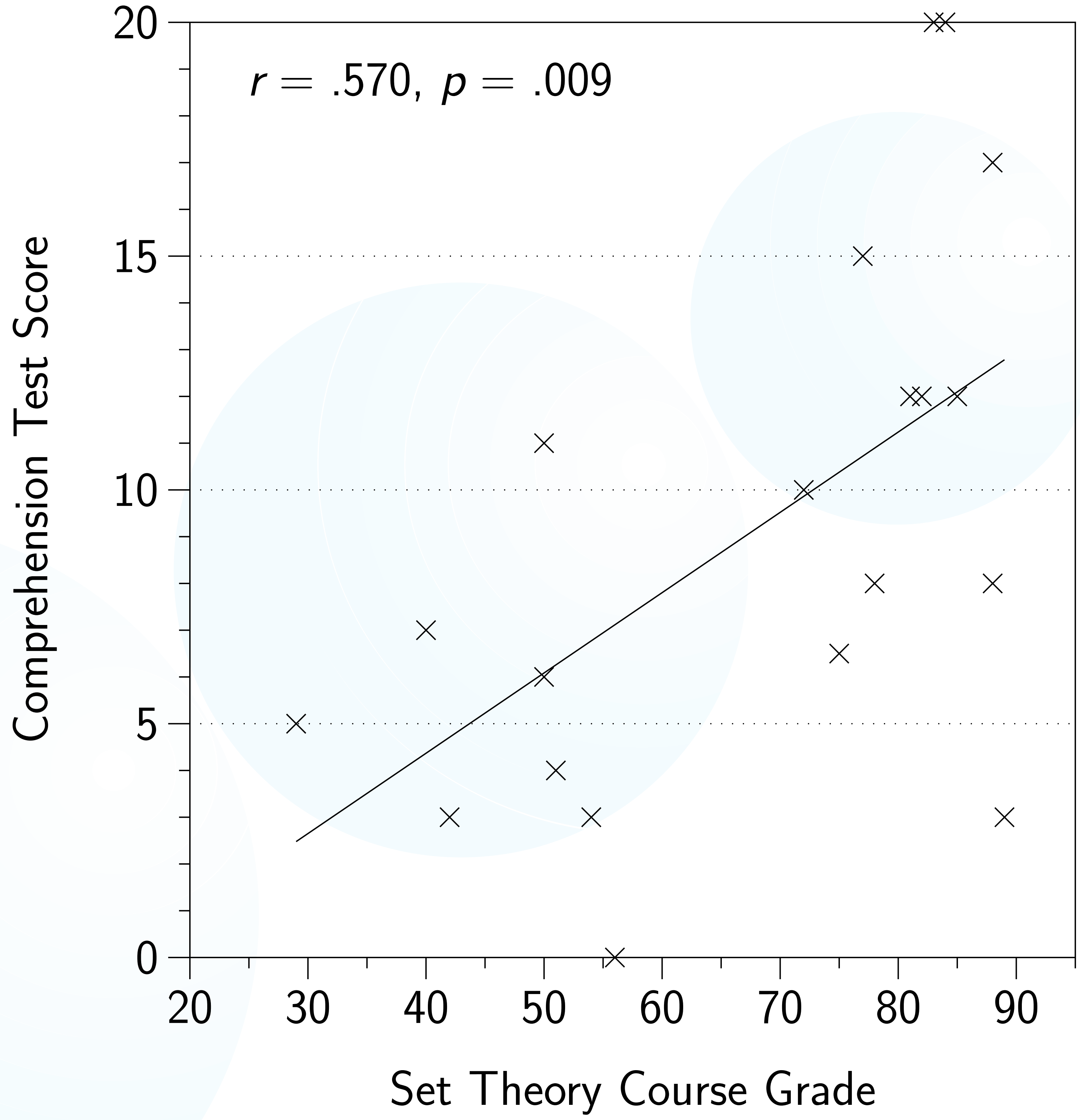

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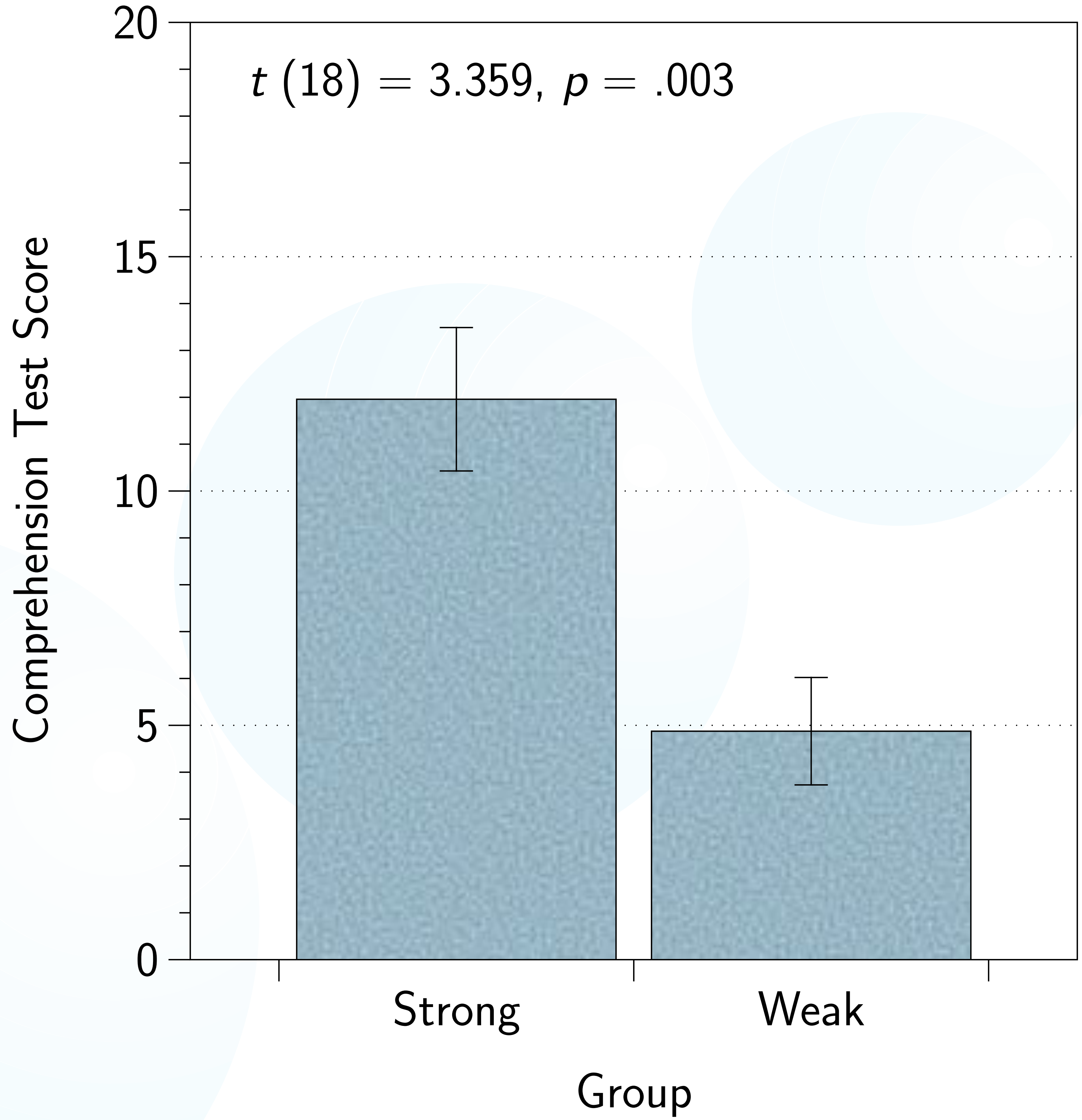

Q1: Learning Differences

Did the two groups learn different amounts?

Yes, the ‘strong’ group learnt significantly more from the text than the weak group.

Q2: Counterintuitive Ideas

- Was this difference due to different ways of approaching the ‘counterintuitive’ sections of the text?
- The conceptual change account suggests that the weaker students would find these areas more confusing and harder to process than the stronger students.
- Traditional way to measure processing difficulty is by looking at **mean fixation duration** on a given area (e.g. Just & Carpenter, 1976).
- Longer mean fixations on a given area mean that it was harder to process.

Q2: Counterintuitive Ideas

Now let's make some explicit calculations with cardinals. Because a countable union of countable sets is countable, we find that

$$n + \aleph_0 = \aleph_0 + n = \aleph_0, \text{ for any finite cardinal } n,$$

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This shows us that we have no possibility of defining subtraction of cardinals where infinite cardinals are involved, for what would $\aleph_0 - \aleph_0$ be? According to the above results it could be any finite cardinal or $\aleph_0$ itself, so subtraction cannot be defined

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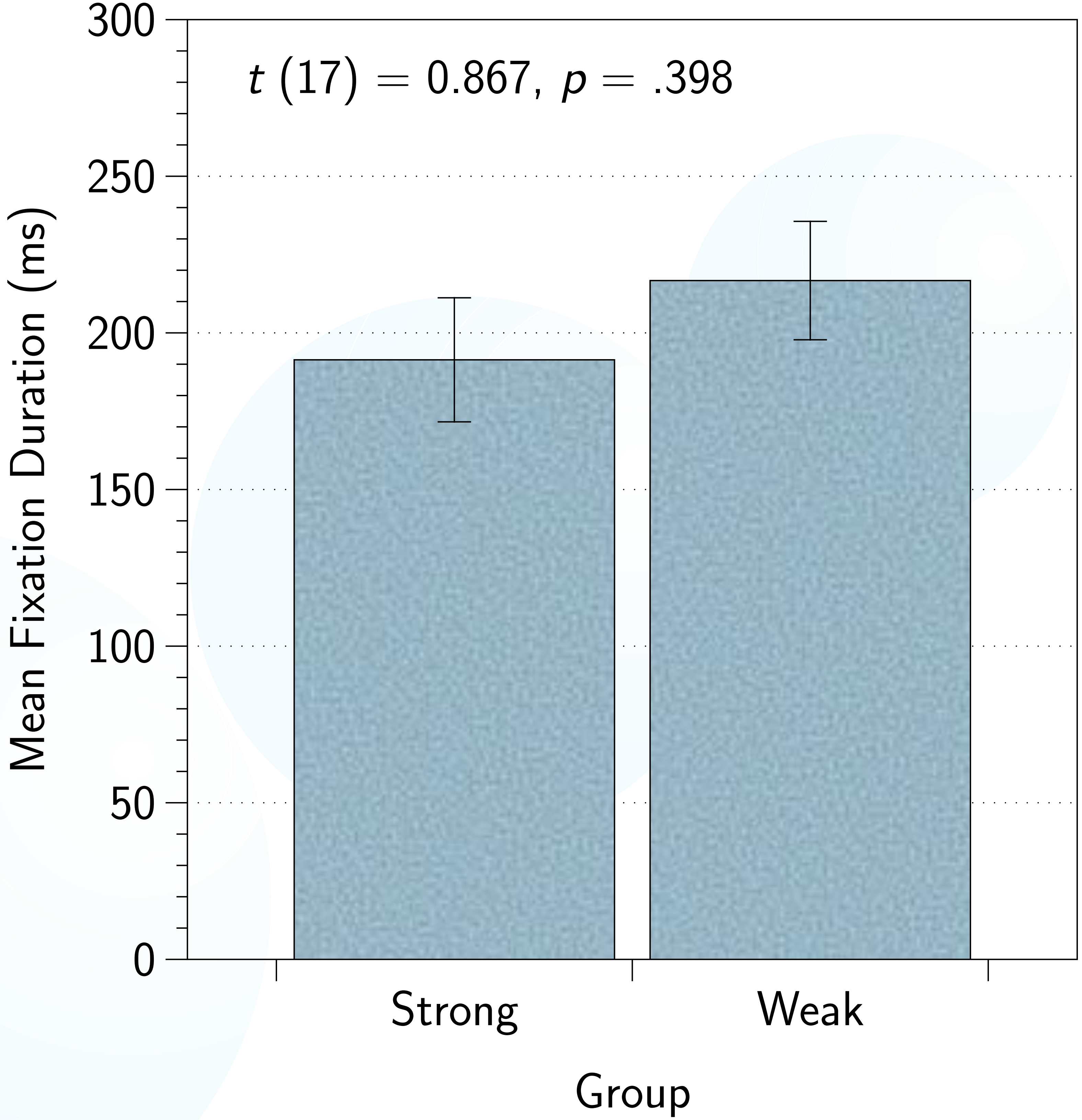

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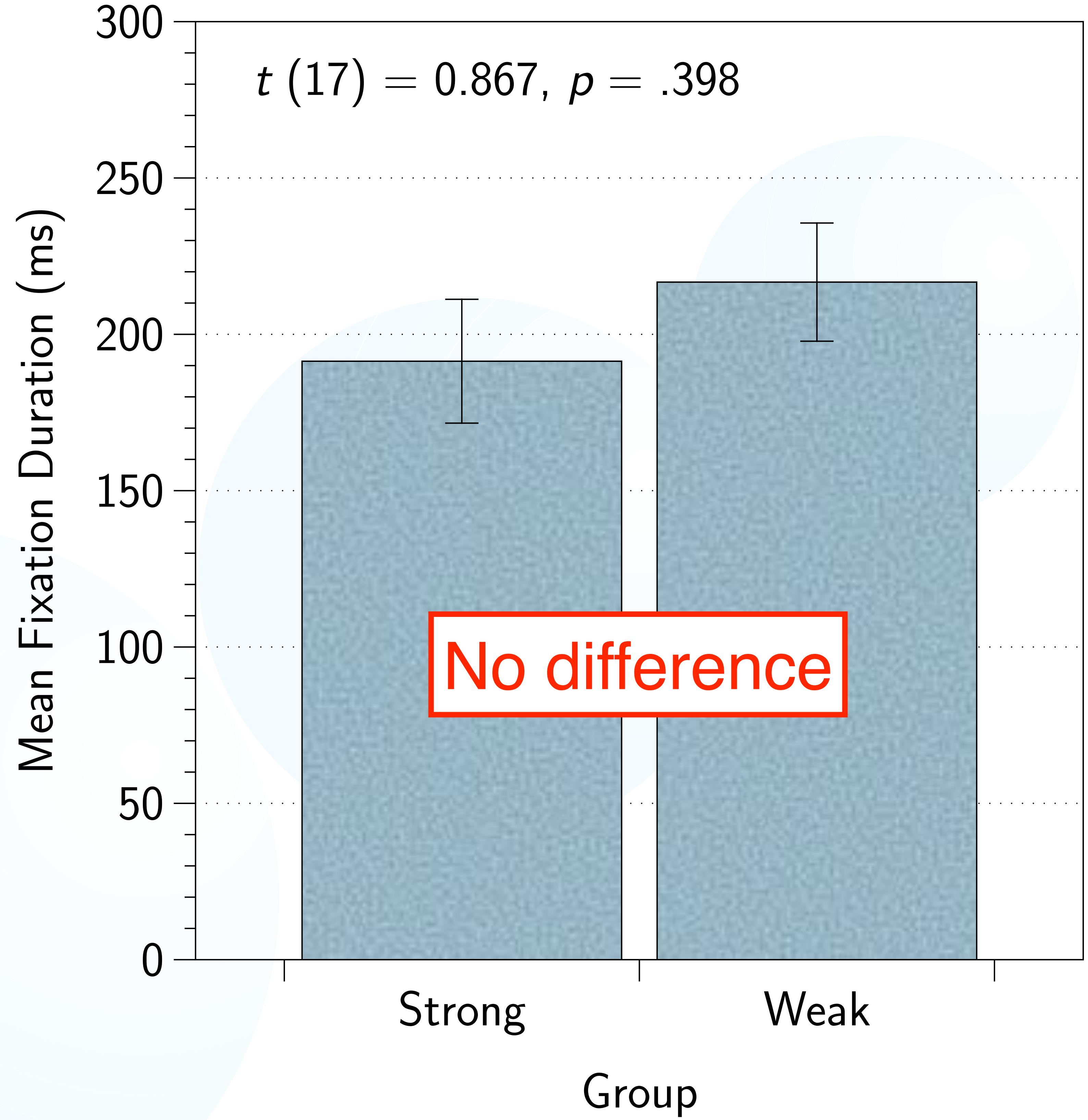

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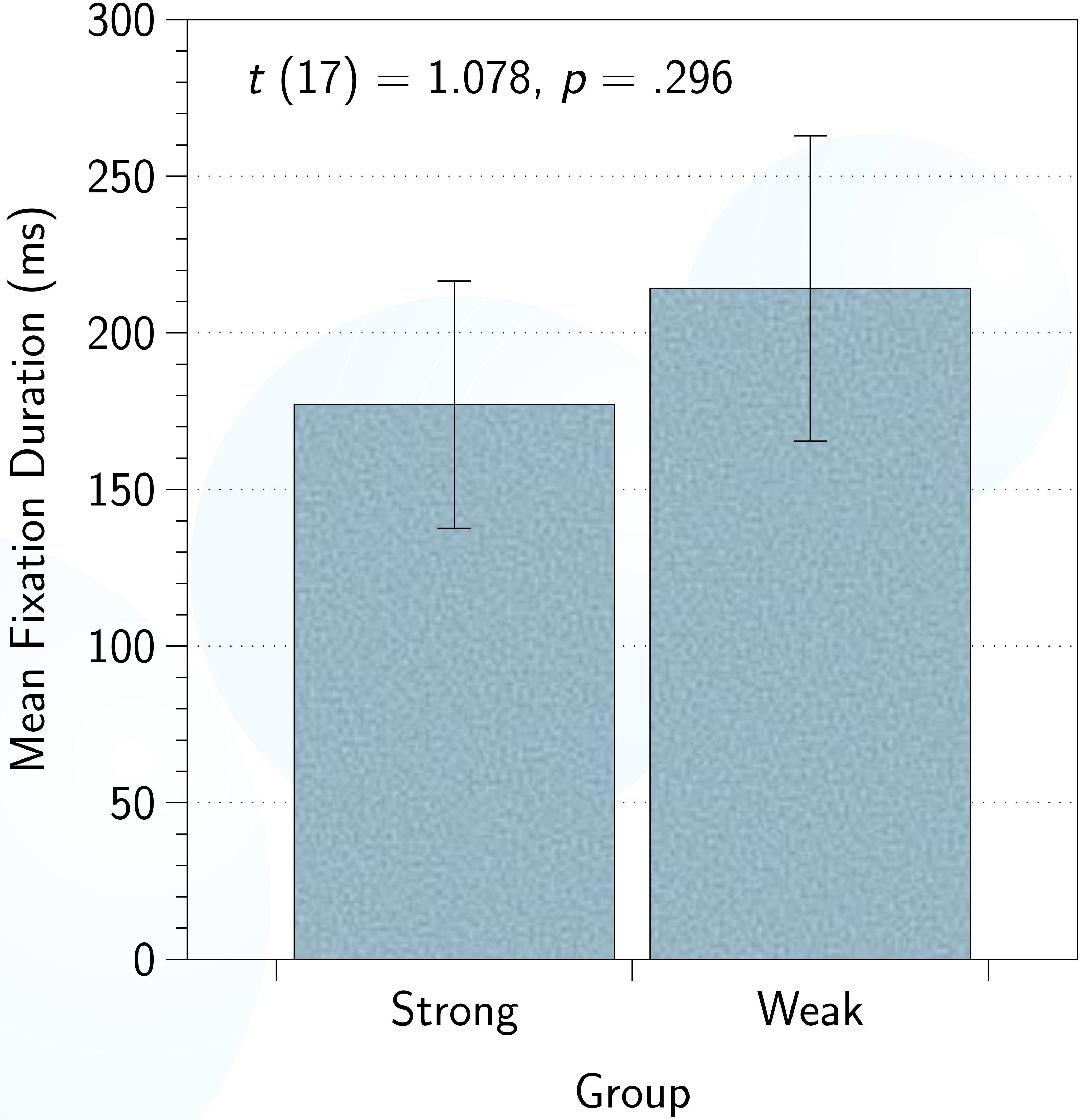

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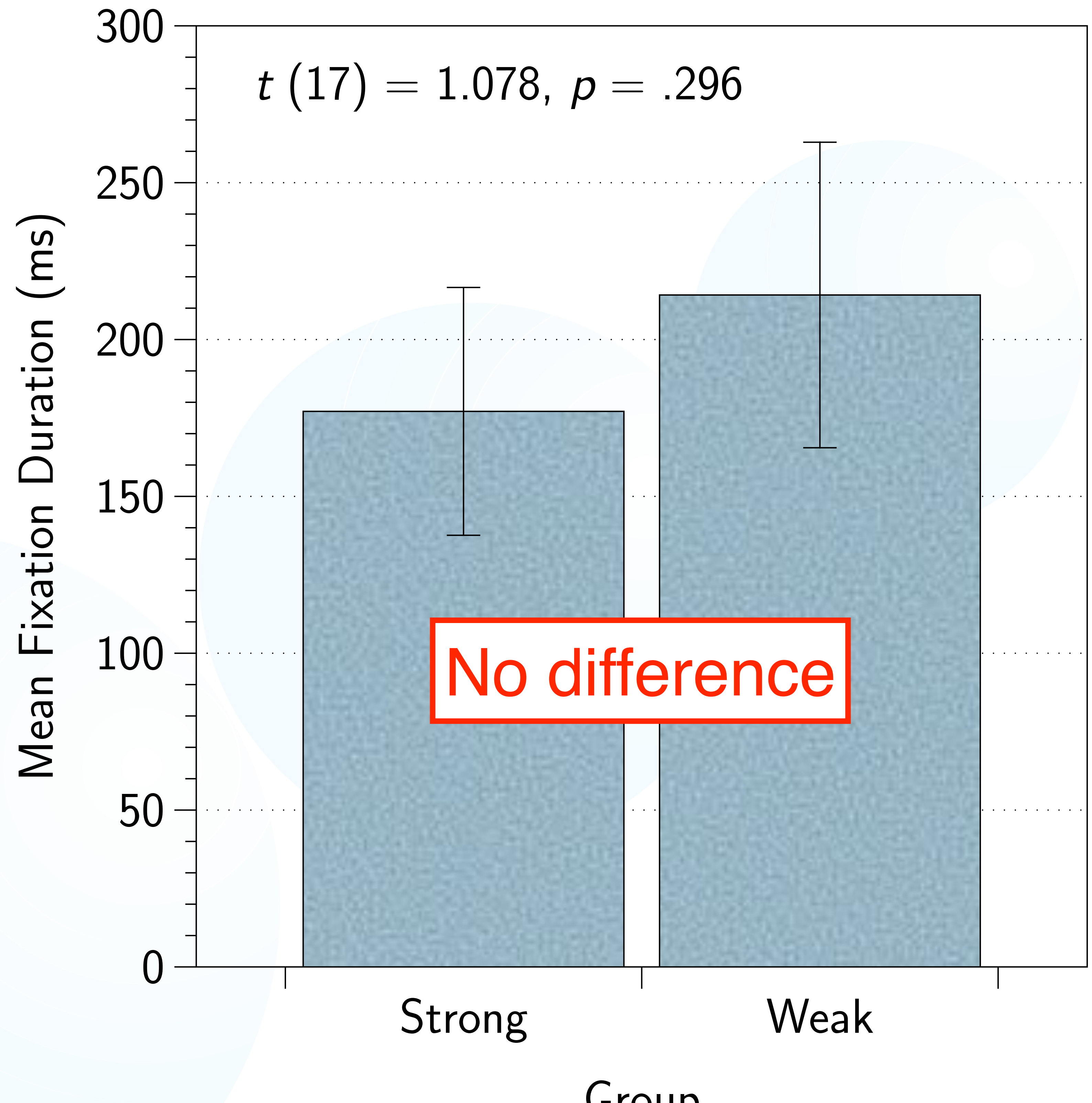

Q2: Counterintuitive Ideas

- No differences between the groups in processing difficulty associated with counterintuitive areas.
- (NB. if you control for individual differences in overall fixation duration, then there are still no differences)
- Also no differences in raw measure of dwell time spent studying these areas, or number of returns to these areas.

Q2: Counterintuitive Ideas

Was this difference due to different ways of approaching the ‘counterintuitive’ sections of the text?

No evidence in favour of this hypothesis.

Apparently the two groups found the difficulty of these sections to be roughly similar.

Q3: What else?

- If the difficulty was not due to difficulty accommodating counterintuitive knowledge, as predicted by the conceptual change account, why did the weak group learn so much less?
- We looked at where participants focused their attention throughout the text.
- Brief highlights here.

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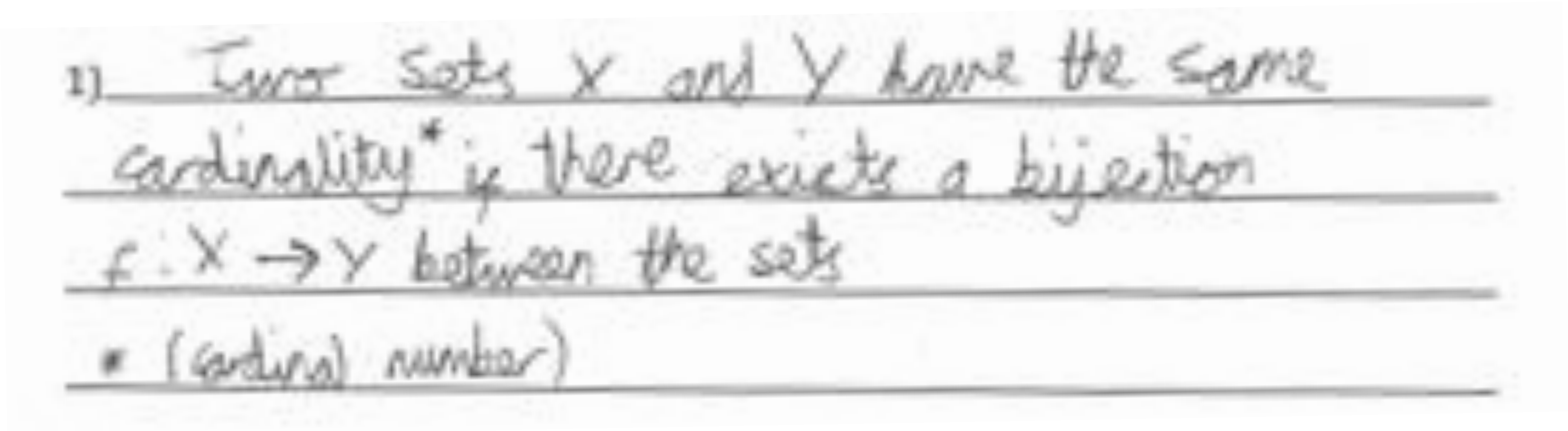

1) Two sets X and Y have the same cardinality* if there exists a bijection $f: X \to Y$ between the sets

* (cardinal number)

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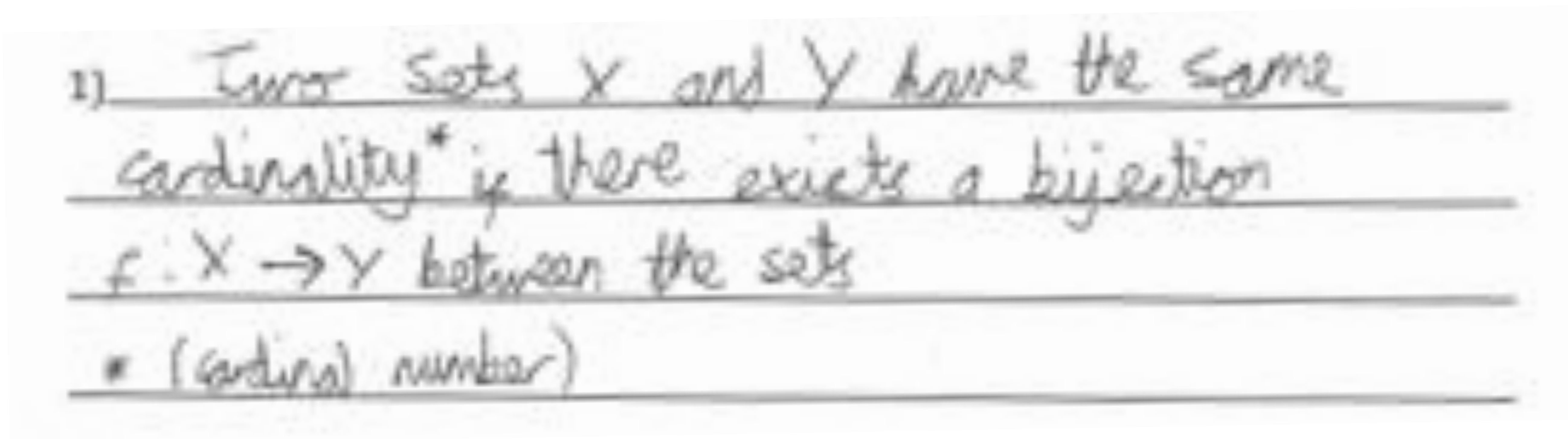

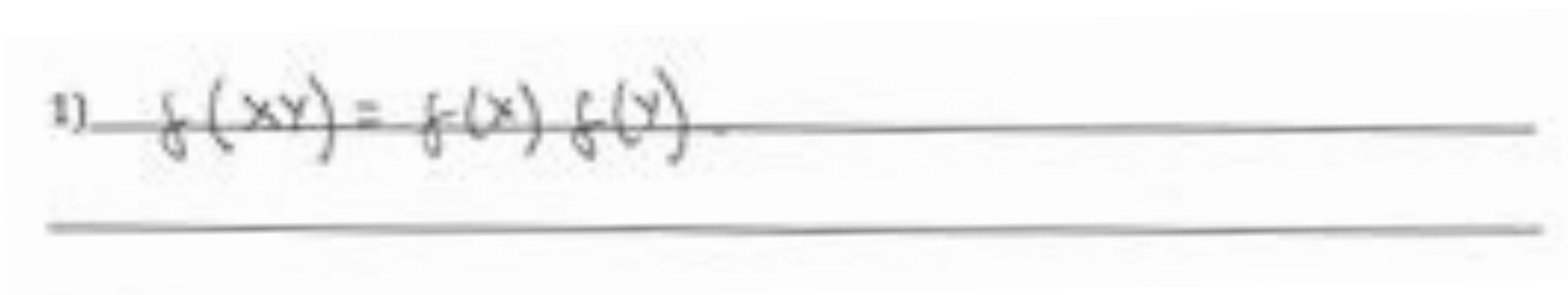

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Rather than begin with the full hierarchy of infinities, let's begin with what turns out to be the smallest of them. The standard set for comparison purposes we'll take to be the natural numbers $\mathbb{N}$. It is useful to consider $\mathbb{N}$ rather than $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ simply because a bijection $f : \mathbb{N} \to B$ organises the elements of B into a sequence; we can call $f(1)$ the *first* element of B using this bijection, $f(2)$ the *second*, and so on... Using this process we set up a method of *counting* B. Of course, if we actually say the elements one after another using this bijection, '$f(1), f(2), \ldots$', we never actually reach the end, but we do know that given any element $b \in B$, then $b = f(n)$ for some sense, this is correct; one of the triumphs of set theory is that the concept of infinity can be given a clear interpretation. We find not one infinity, but many, a vast hierarchy of infinities. We can answer a question like ‘How many rational numbers are there?’, with the surprising reply ‘as many as there are natural numbers’. The most useful type of question is exemplified by this answer. Rather than ask ‘how many’ elements there are in a given set, it is much more profitable to compare two sets and ask if there are as many elements in the two of them. This can be described by saying that there are ‘the same number of elements’ in sets A and B if there is a bijection $f : A \to B$.

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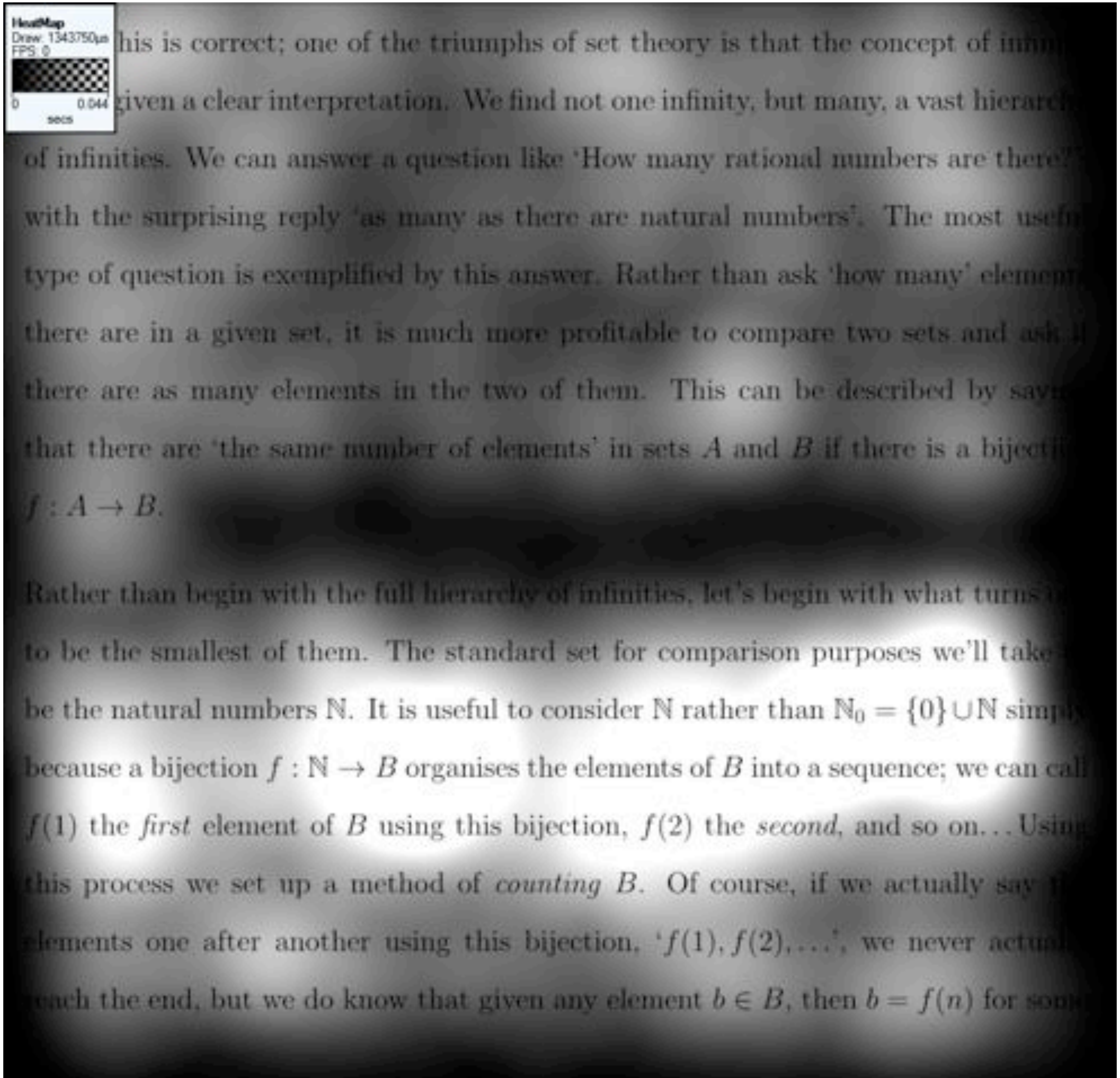
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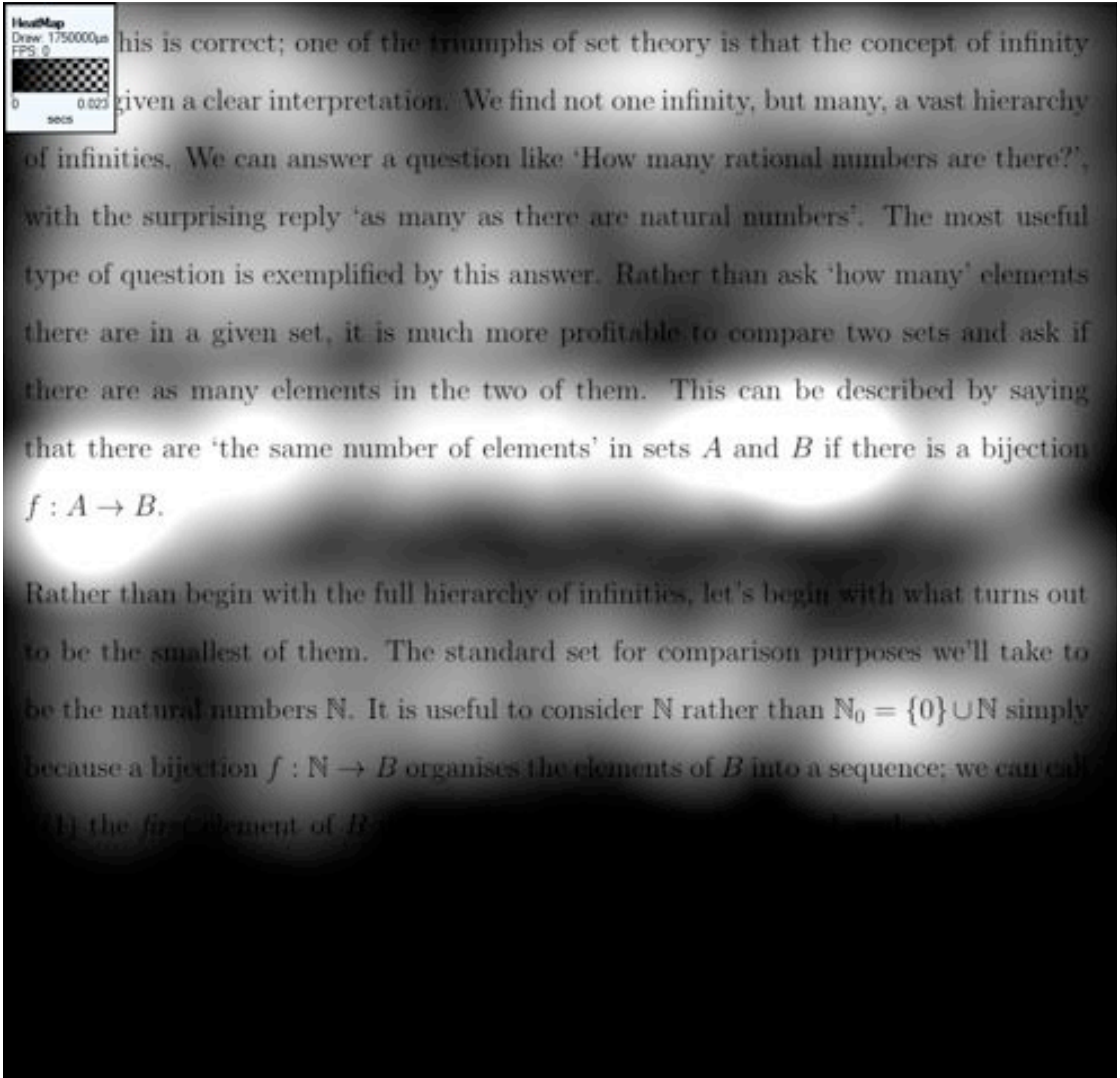

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Definitions

- Of course the $f(XY) = f(X)f(Y)$ person wasn't going to be able to answer this question: they didn't attend to the definition.
- If they didn't know what $|X| = |Y|$ meant, the rest of the chapter would have been jibberish.
- Was this lack of focus on definitions the case in general?

Cardinal Arithmetic

Just as we can add, multiply and take powers of finite cardinal numbers, we can mimic the set-theoretic procedures involved and define corresponding operations on infinite cardinals. Some, but not all, of the properties of ordinary arithmetic carry over to cardinal numbers, and it is most instructive to see which ones. First of all the definitions:

Addition. Given two cardinal numbers α, λ (finite or infinite), select disjoint sets A, B such that $|A| = \alpha$, $|B| = \lambda$. Define $\alpha + \lambda$ to be the cardinal number of $A \cup B$.

Multiplication. If $\alpha = |A|$, $\lambda = |B|$, then $\alpha\lambda = |A \times B|$.

Powers. If $\alpha = |A|$, $\lambda = |B|$, then $\alpha^\lambda = |A^B|$ where A^B is the set of all functions from B to A.

The reader should pause briefly and check that when the sets concerned are finite then this corresponds to the usual arithmetic. In particular, when $|A| = m$ and $|B| = n$, then on defining a function $f : B \to A$, each element $b \in B$ has m possible choices of

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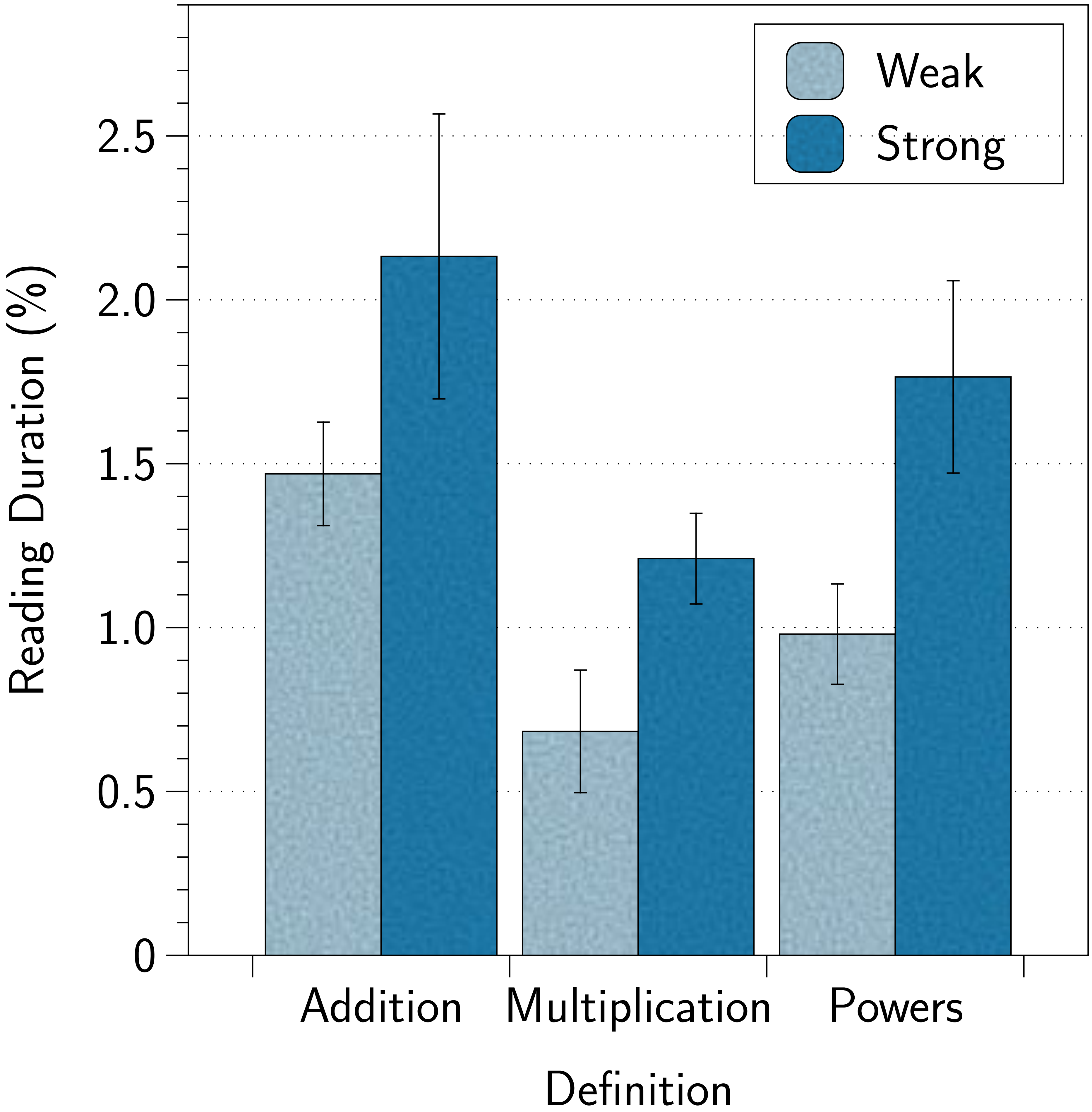
Weak
Strong
Reading Duration (%)
2.5
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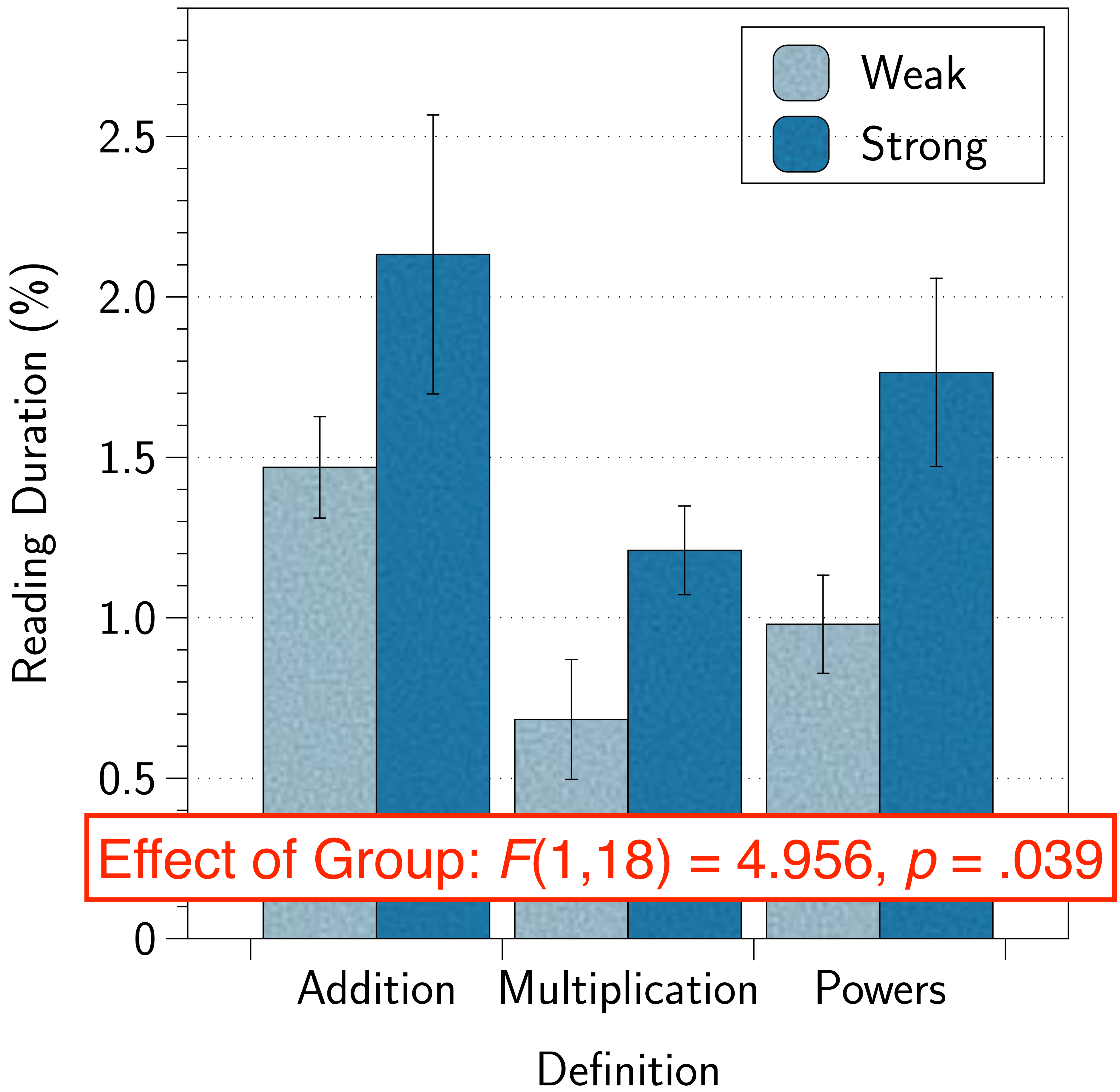

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Effect of Group: F(1,18) = 4.956, p = .039

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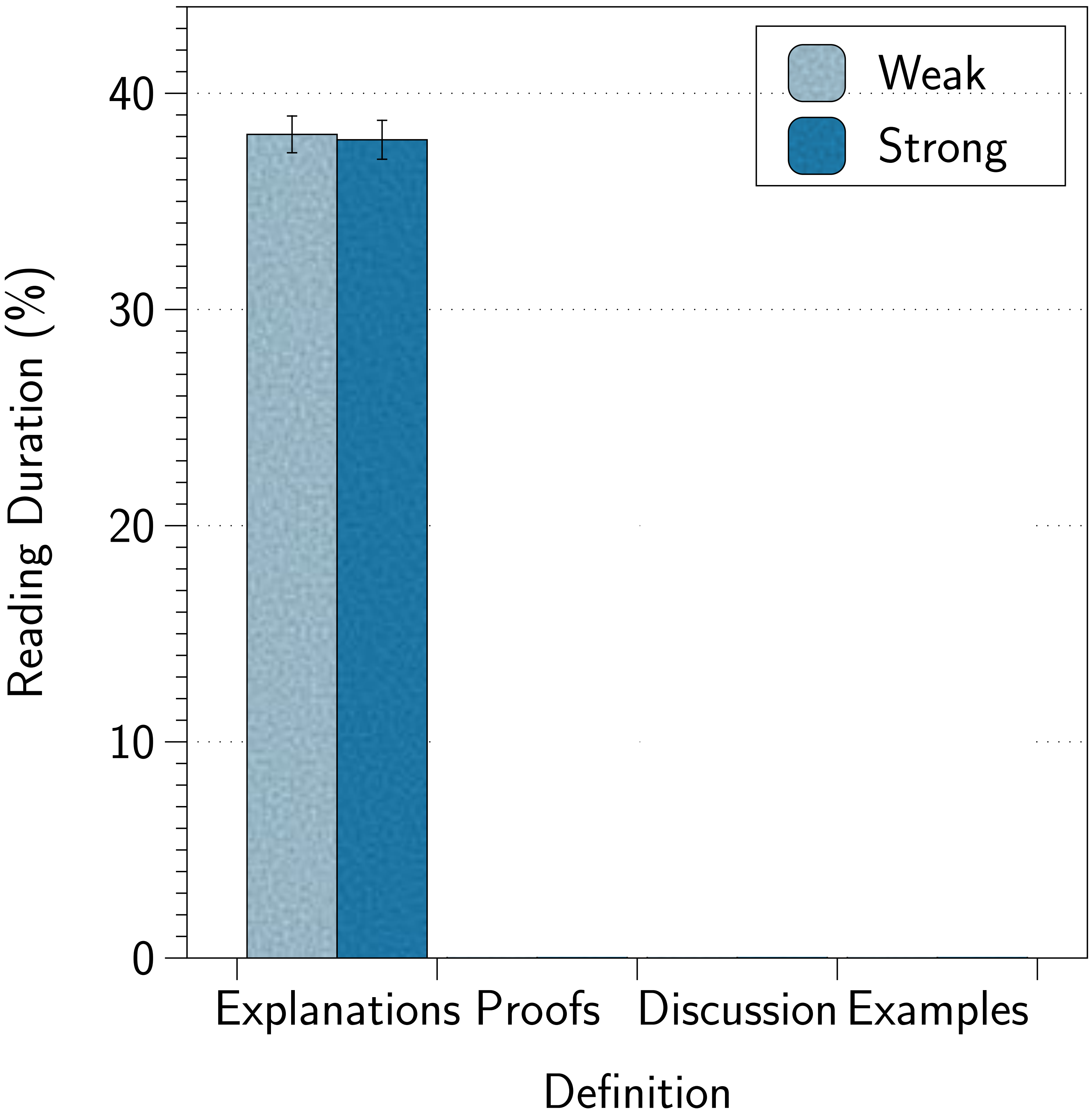

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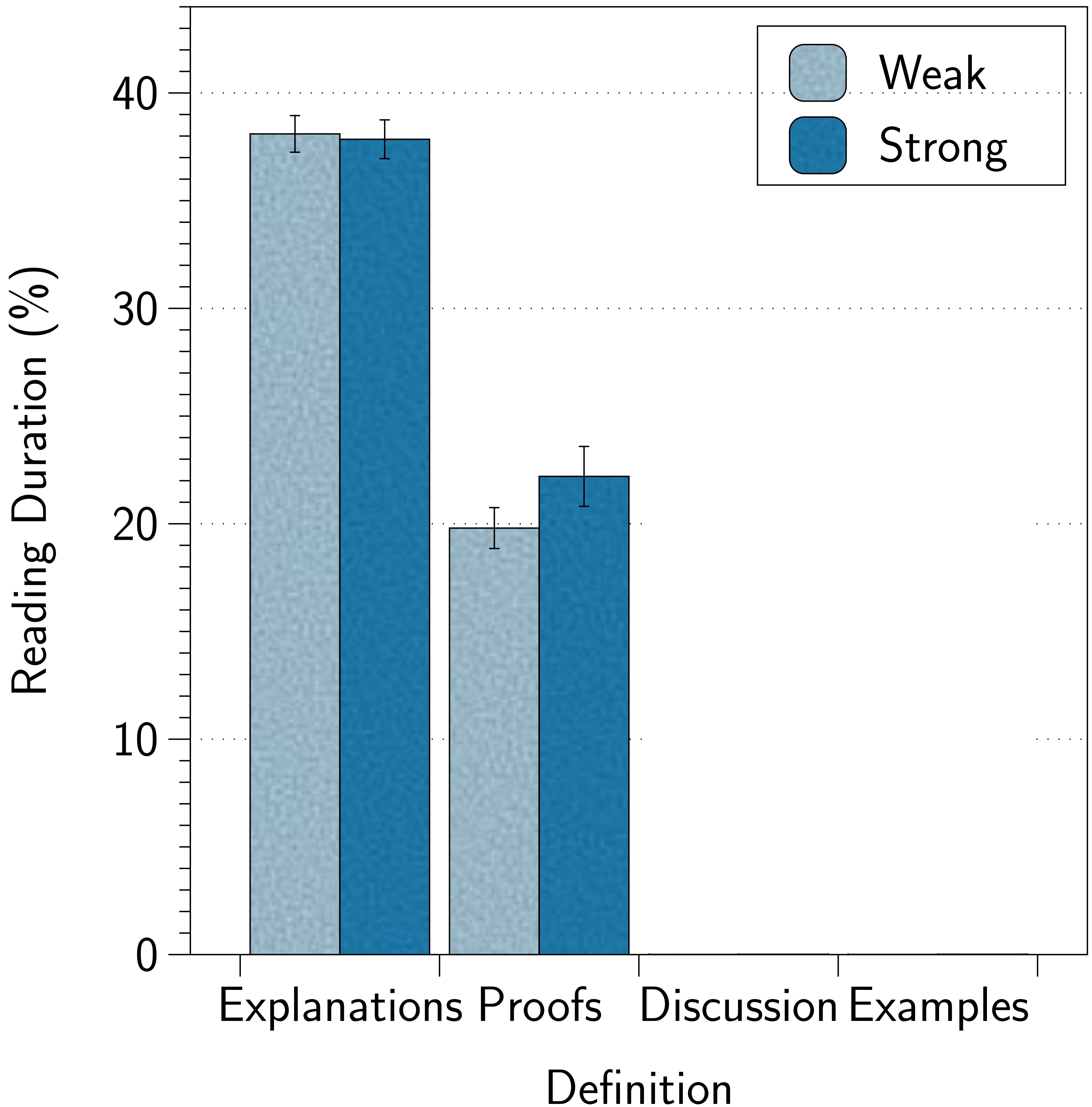

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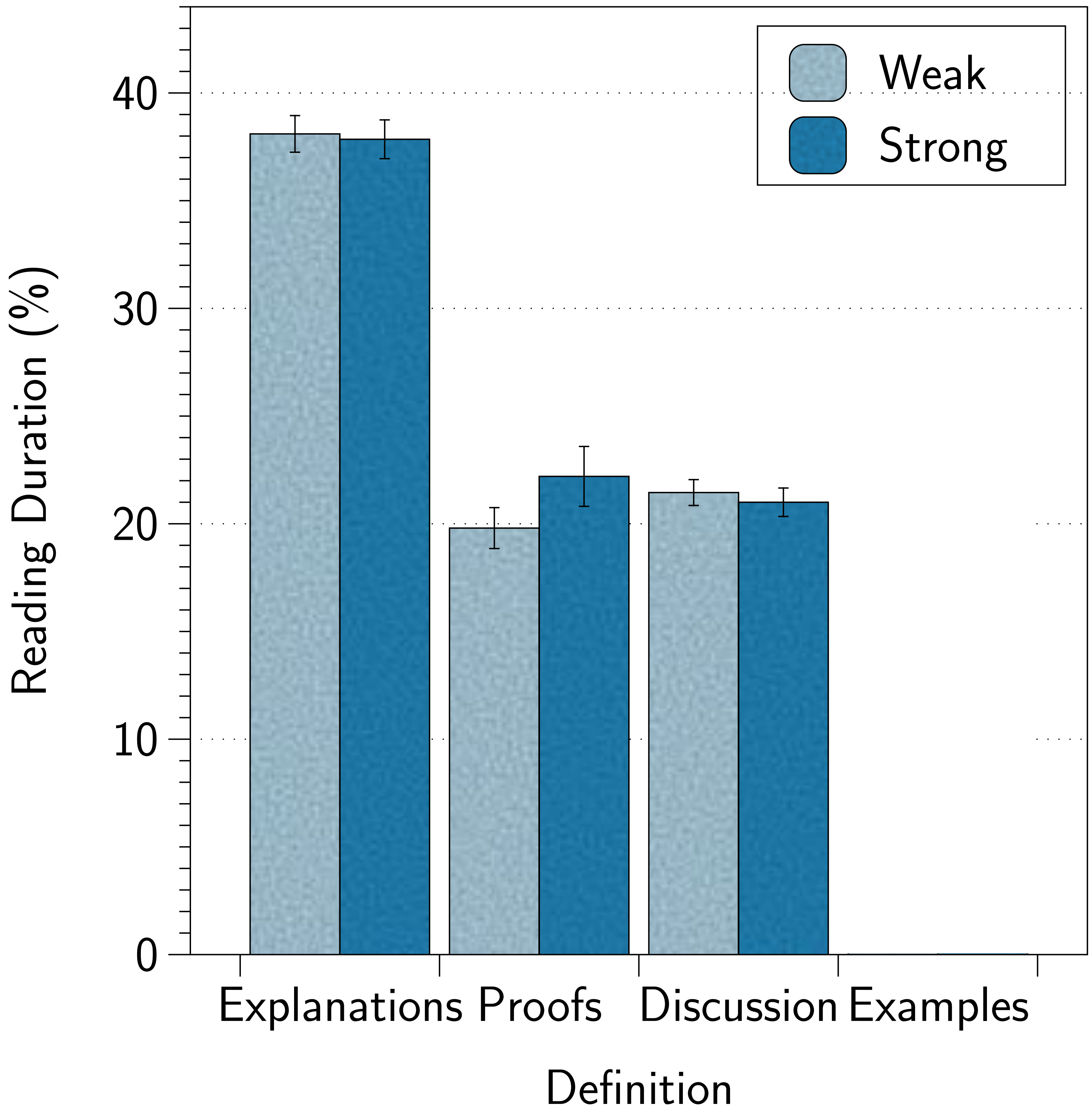

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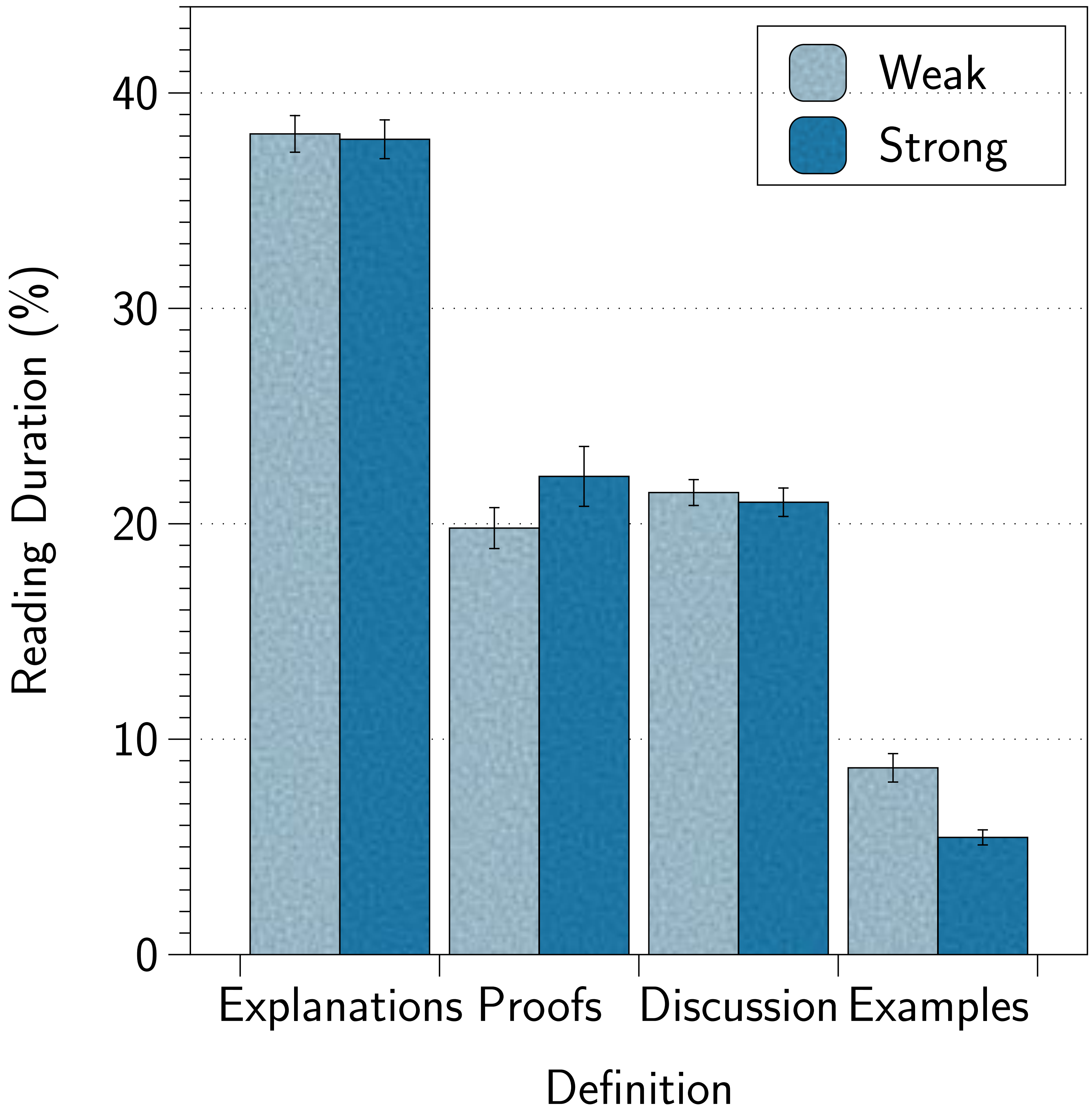

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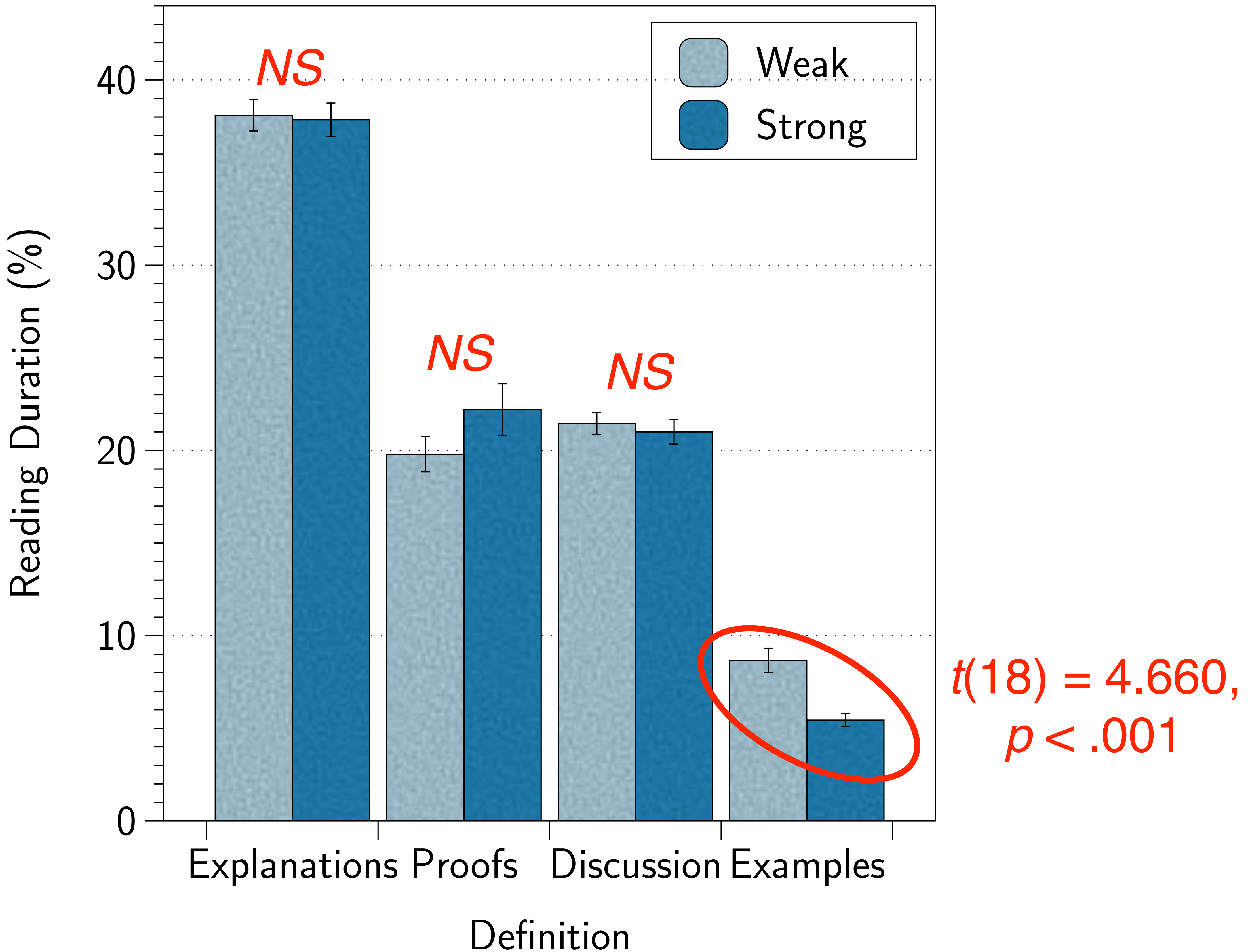

Study 1: Summary

- The difference between strong and weak students was not related to how they dealt with their conflicts with existing knowledge;

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- The weak students had extremely questionable study strategies:
  - They spent around half as long as the strong students reading definitions;
  - and 50% longer reading examples.

Study 2: Improvement?

- If students have ineffective reading strategies, is there a way of improving them?
- Can we simply tell them that they should read more effectively?
- We tried the *Self-Explanation Training* method.

Self-Explanation Effect

Chi et al. (1989) asked students to read material on Newtonian mechanics. Those who did well on problems produced more *self-explanations*: more interpretations of what was read that involved information and relationships beyond those in text.

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Ainsworth & Burcham (2007) distinguished explanations of different quality. Comprehension of a biology text was related to the types of explanation produced.

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- Demonstrate with an example proof;
- Distinguish self-explanation from monitoring and paraphrasing;
- Provide practice reading attempt.

Method

Participants:

- 76 undergraduates (26 first year, 26 second, 24 final).
- Self-explanation and control groups (38 each).

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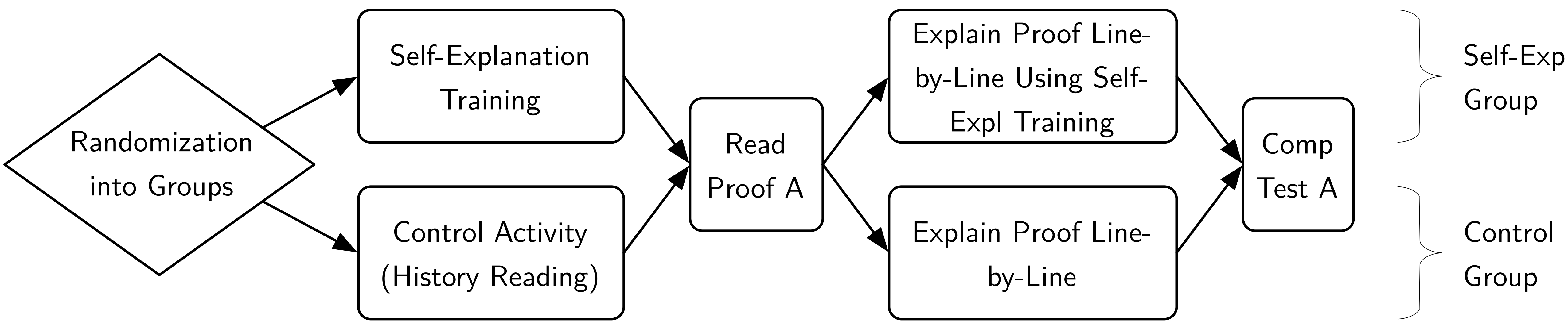
Proof comprehension test:

- Proof that there exist infinitely many triadic primes.
- Comprehension test based on Mejia-Ramos et al.'s (2012) framework.
- Question order randomised; Total possible score 28.

Design


Randomization into Groups
Self-Explanation Training
Control Activity (History Reading)
Read Proof A
Explain Proof Line-by-Line Using Self-Expl Training
Explain Proof Line-by-Line
Comp Test A
Self-Expl Group
Control Group

Explanation Types

Explanations:

- Principle-based: explanation based upon definitions, theorems or ideas not explicit in proof.
- Goal-driven: explanation of how structure relates to goal of text (e.g. proving the theorem).
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This is where the students had problems with the cardinality text

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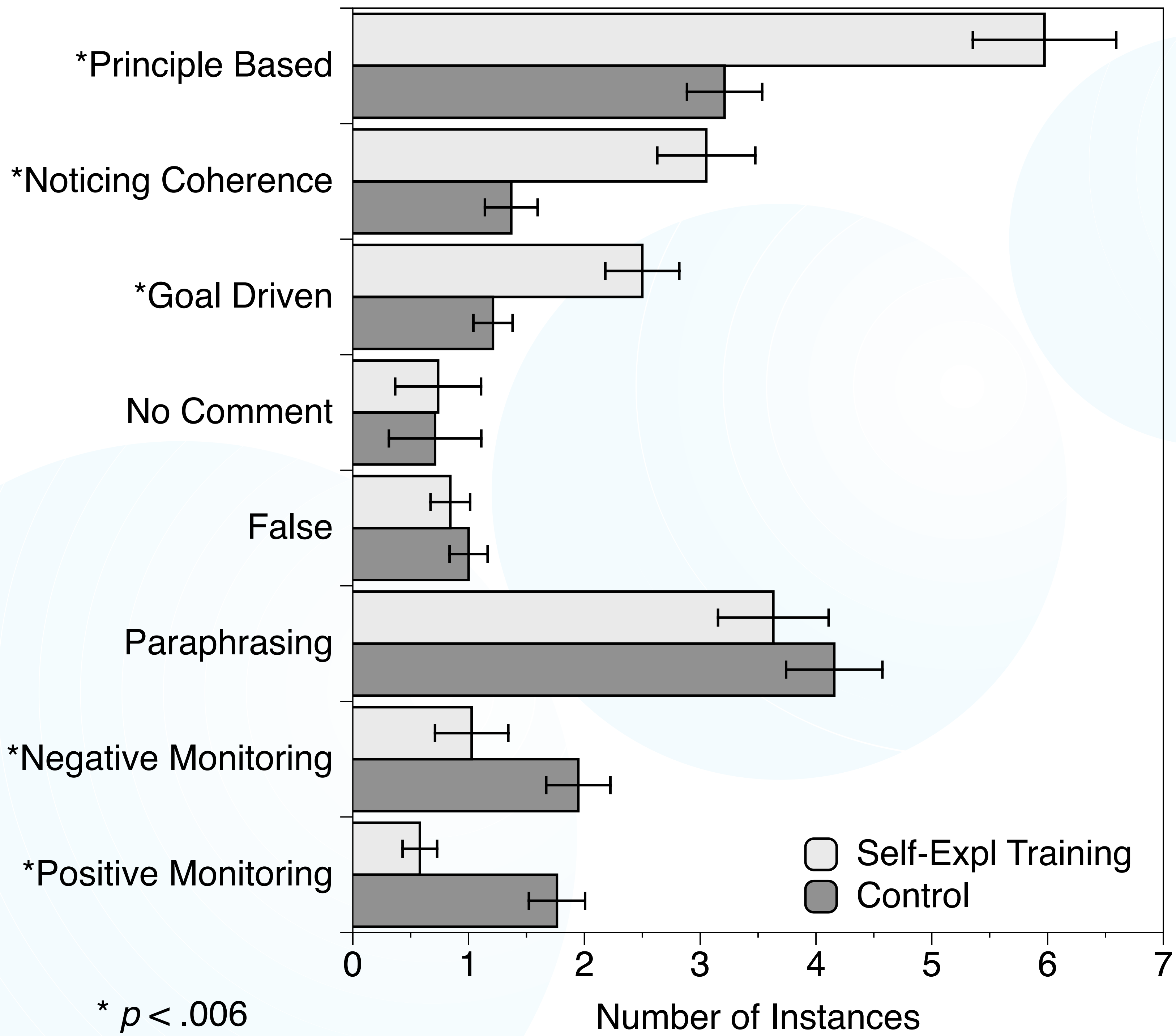
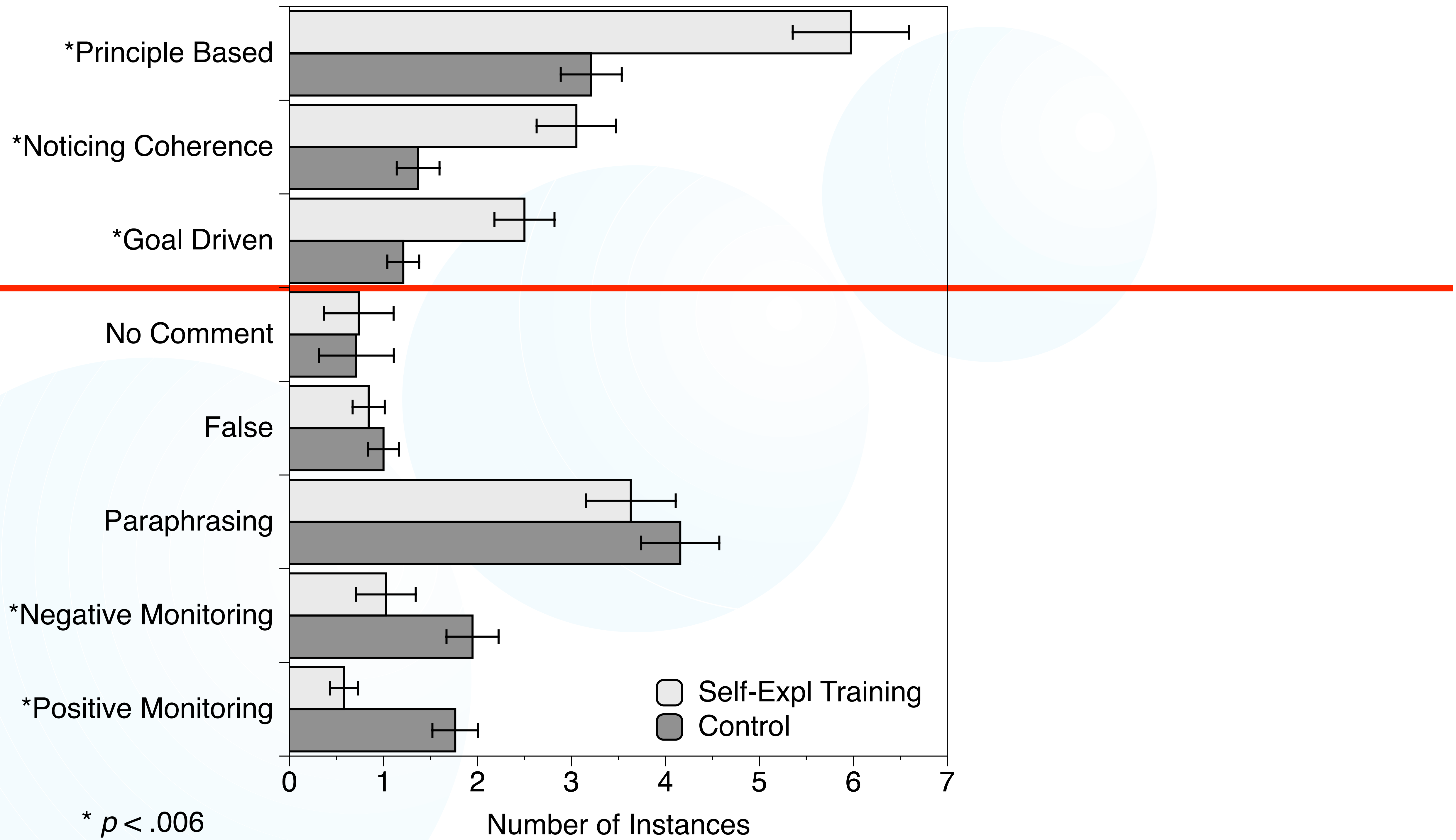


* $p < .006$

Explanation quality



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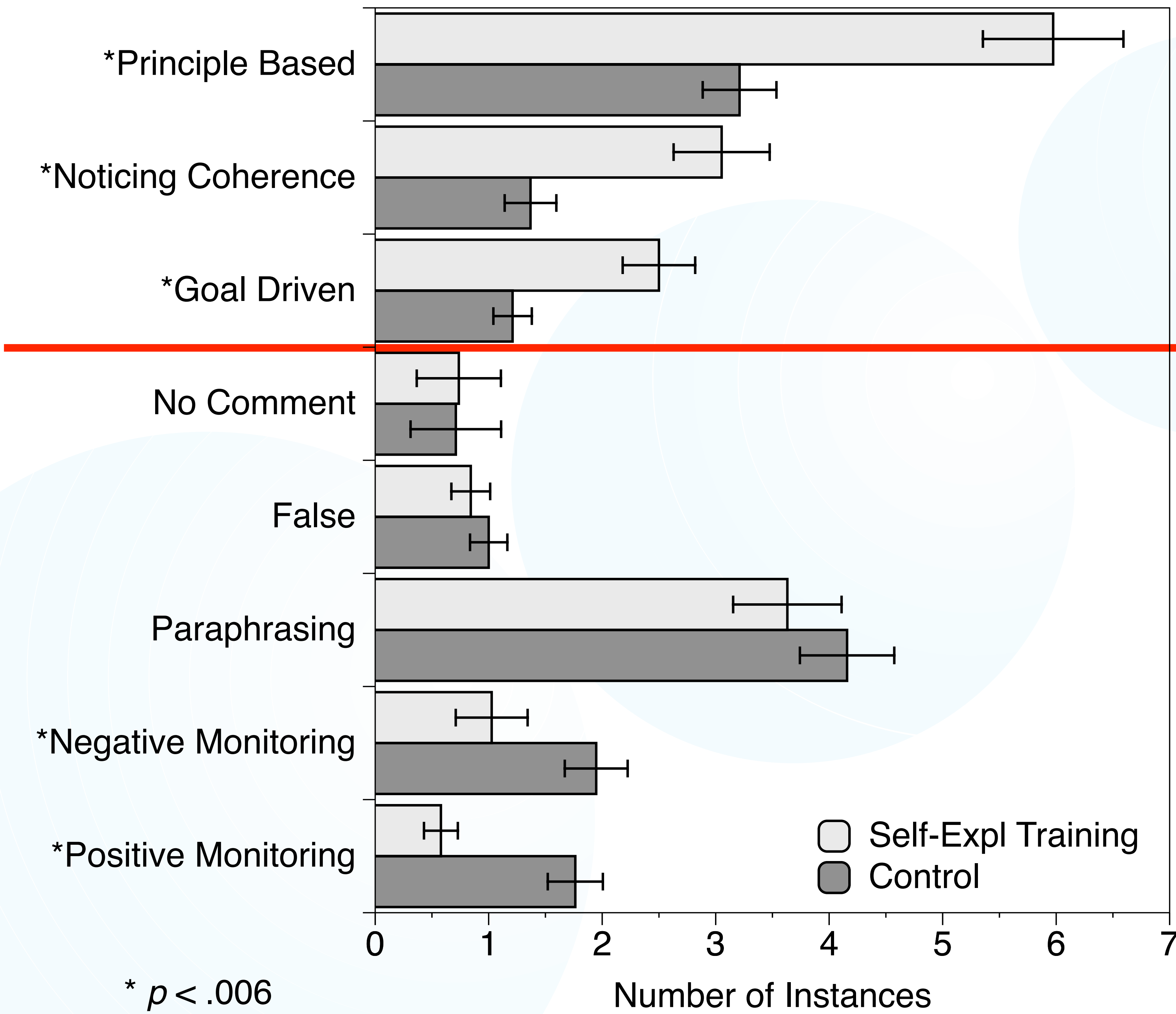

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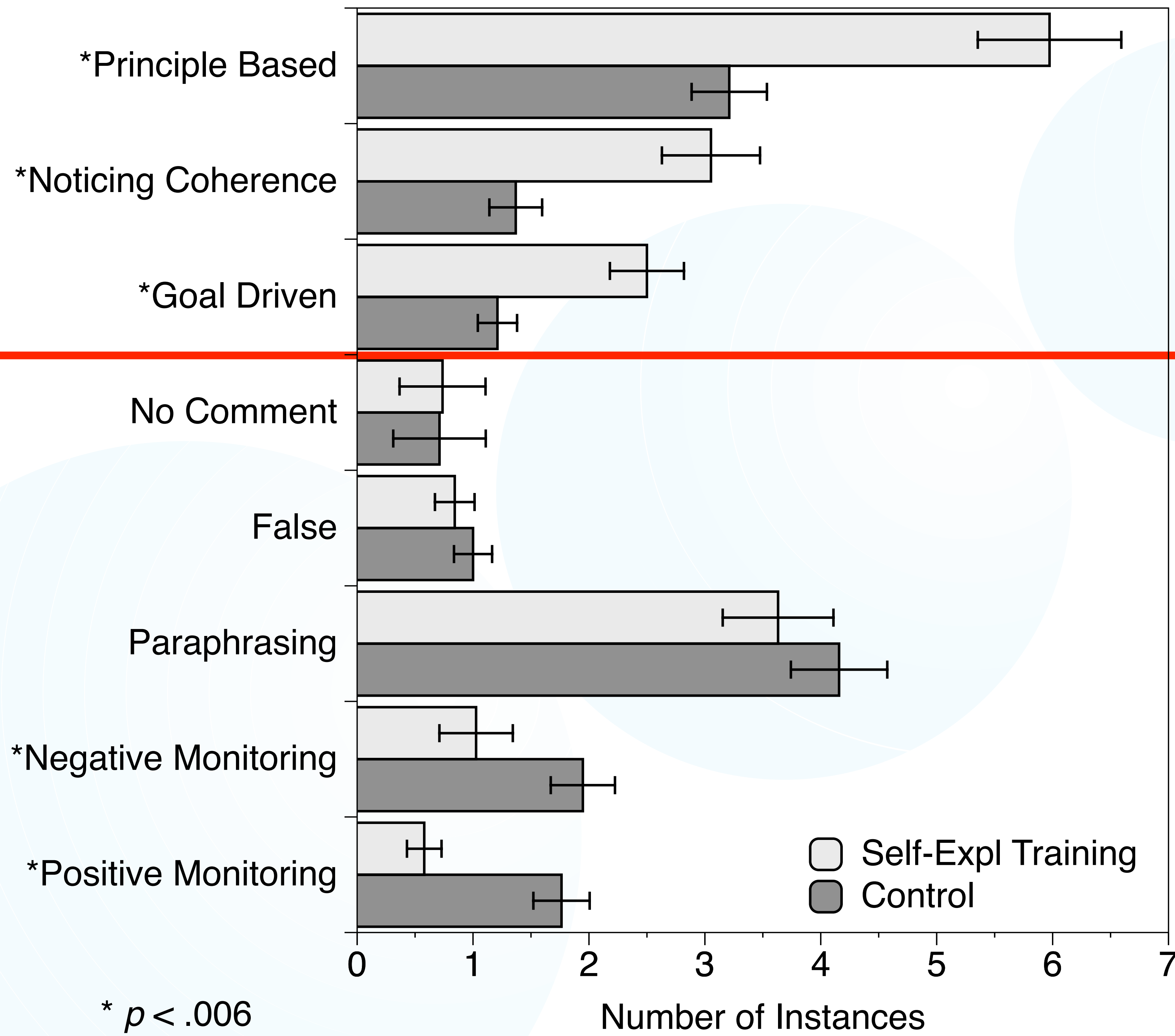

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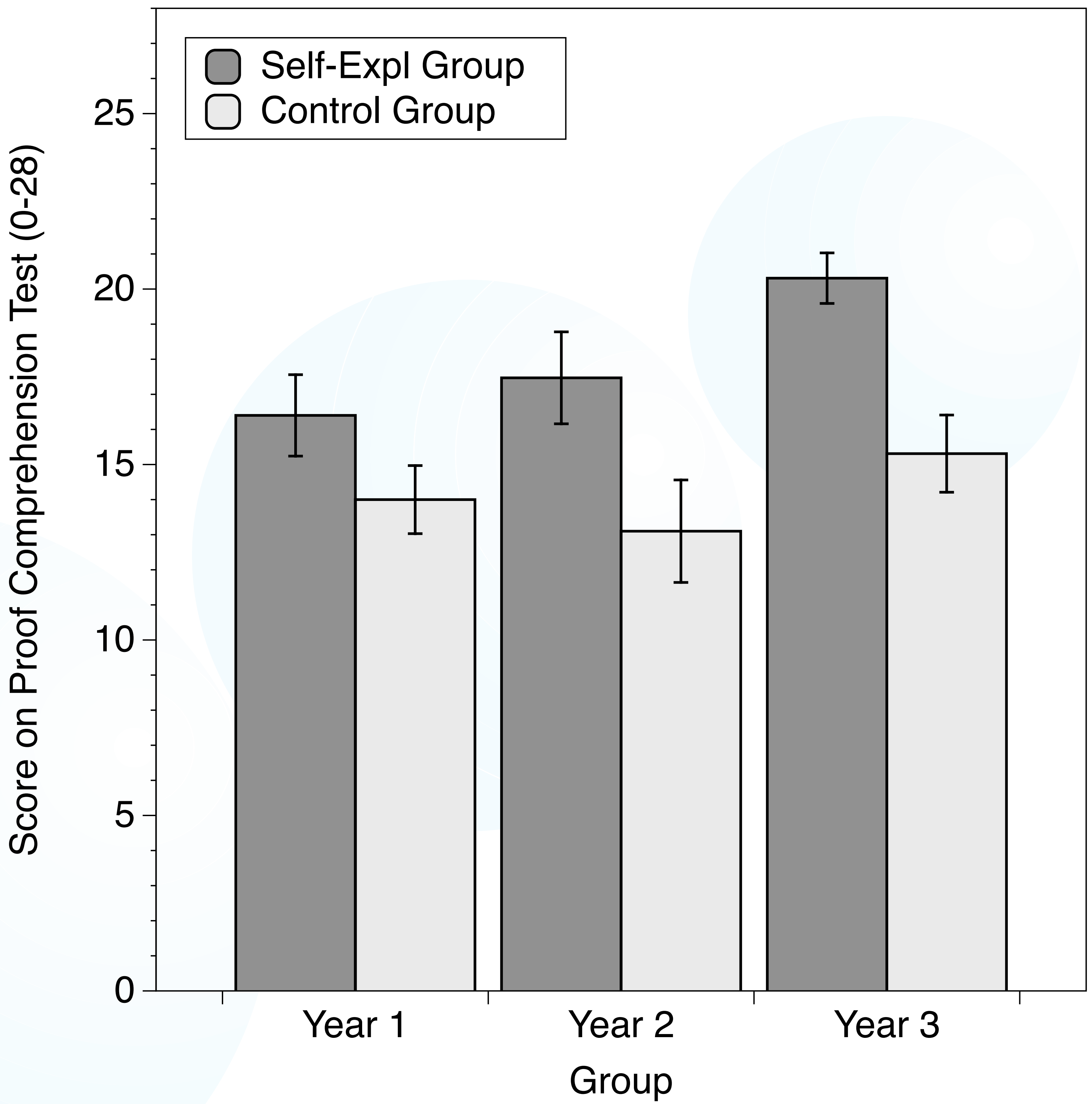
Fewer positive and negative monitoring statements

Comprehension Score



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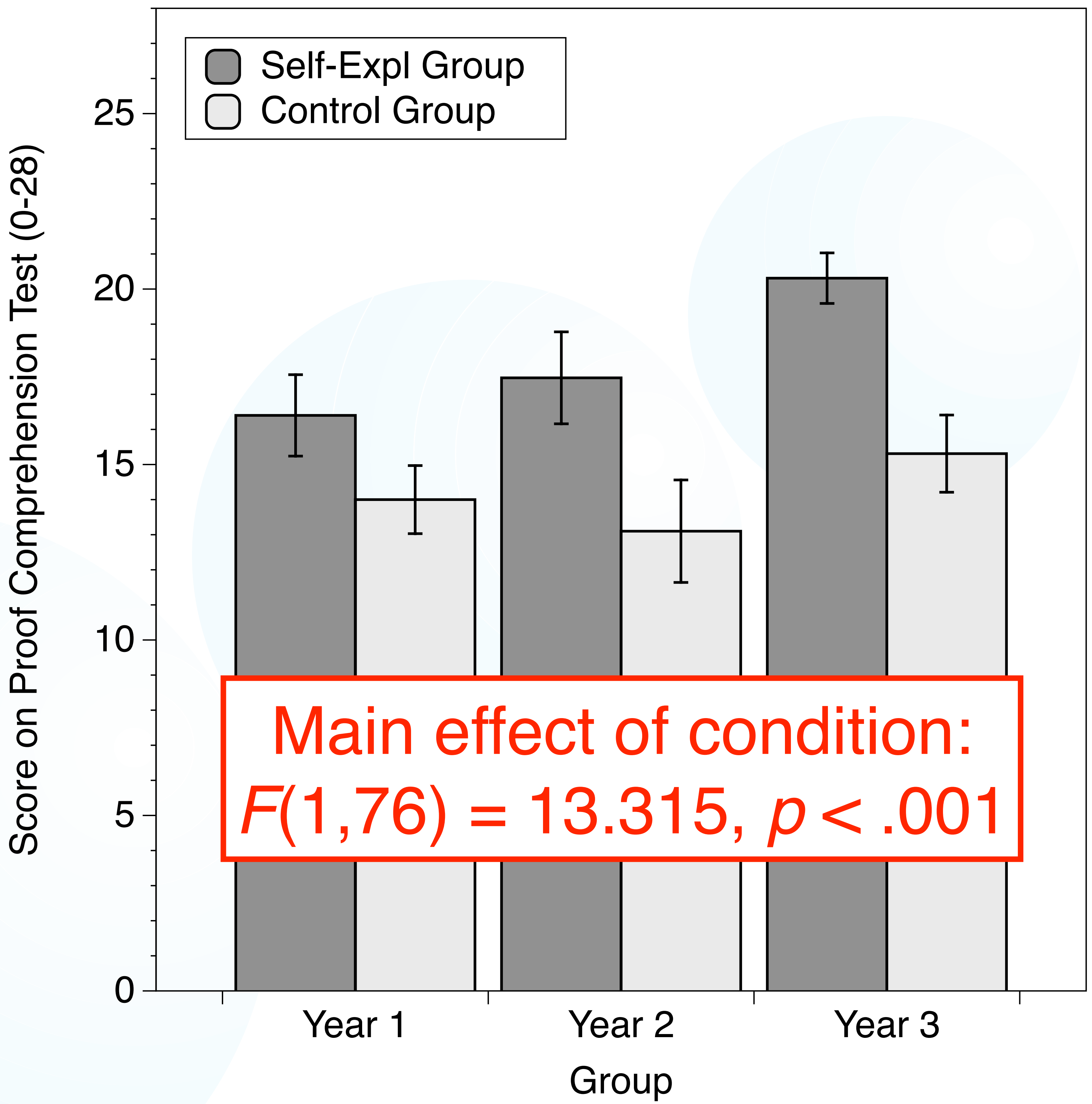

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Average scores:

- Self-explanation group: 18.2 (SD=4.2)
- Control group: 14.2 (SD=4.0)

Effect size: very large, *d*=0.950.

Study 3: Genuine Pedagogy?

Does this technique work in genuine pedagogical settings?

Method

Participants:

- 107 first-year undergraduates; Calculus lectures.
- Self-explanation group (53) and control group (54).

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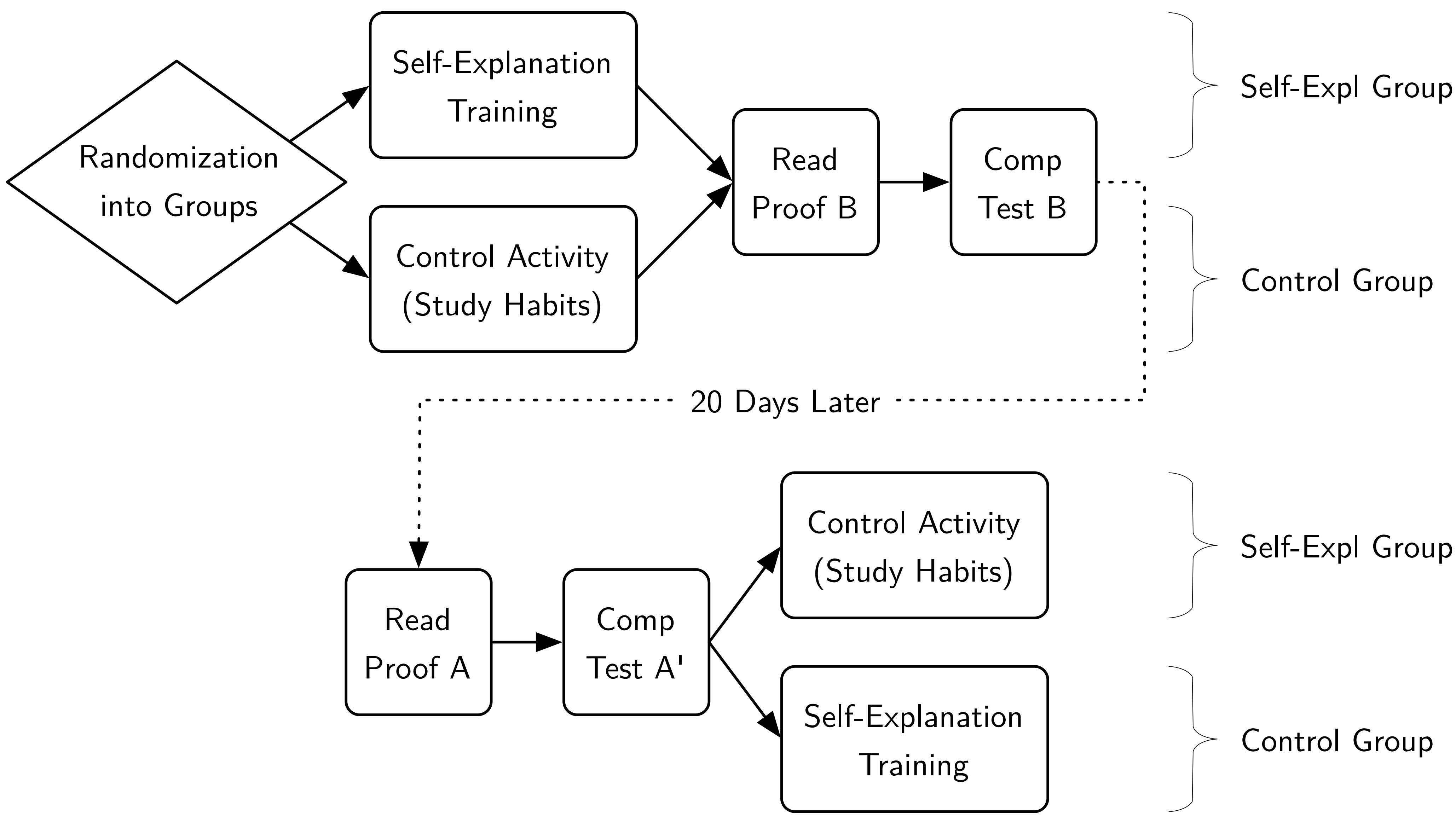
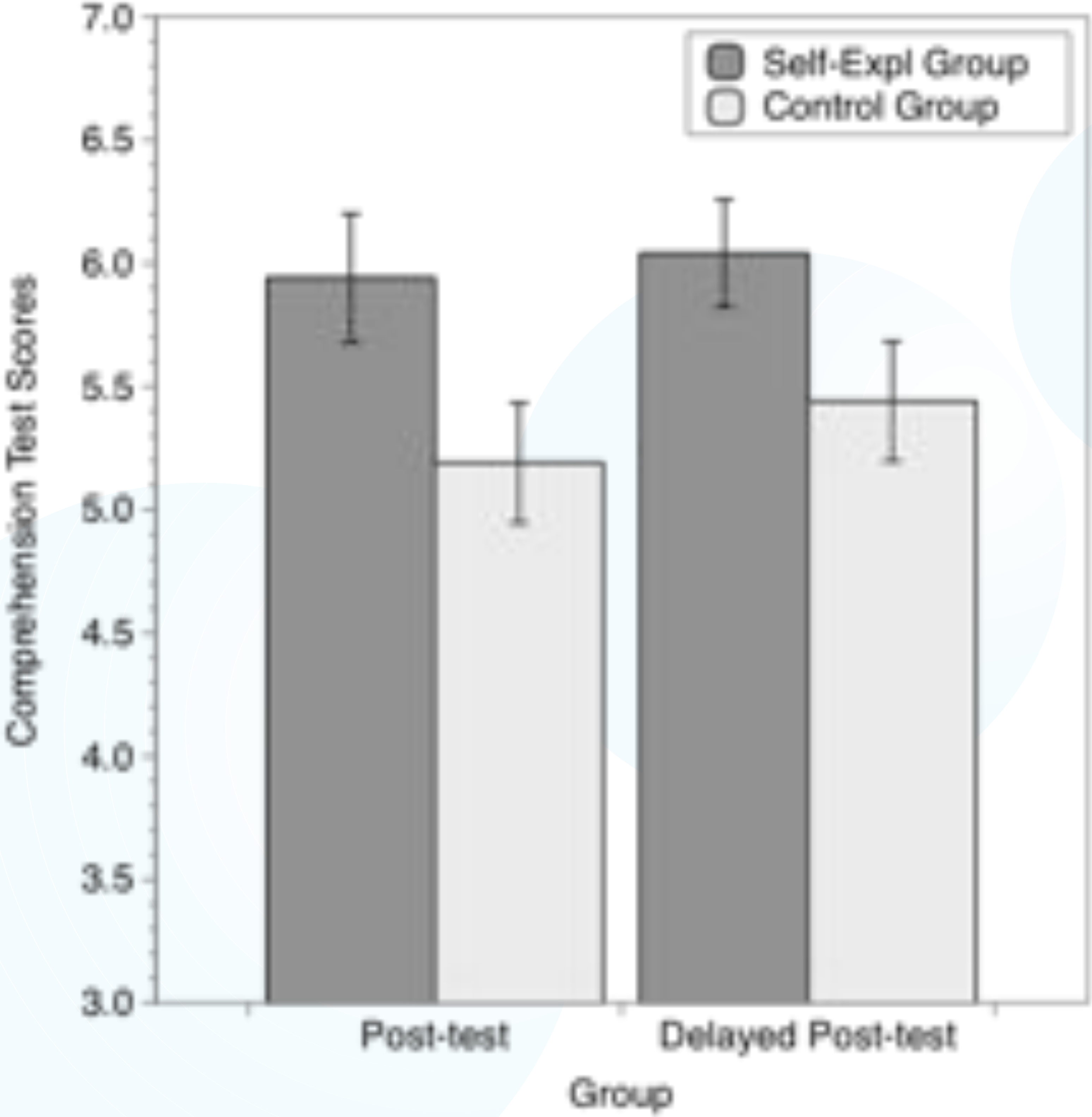
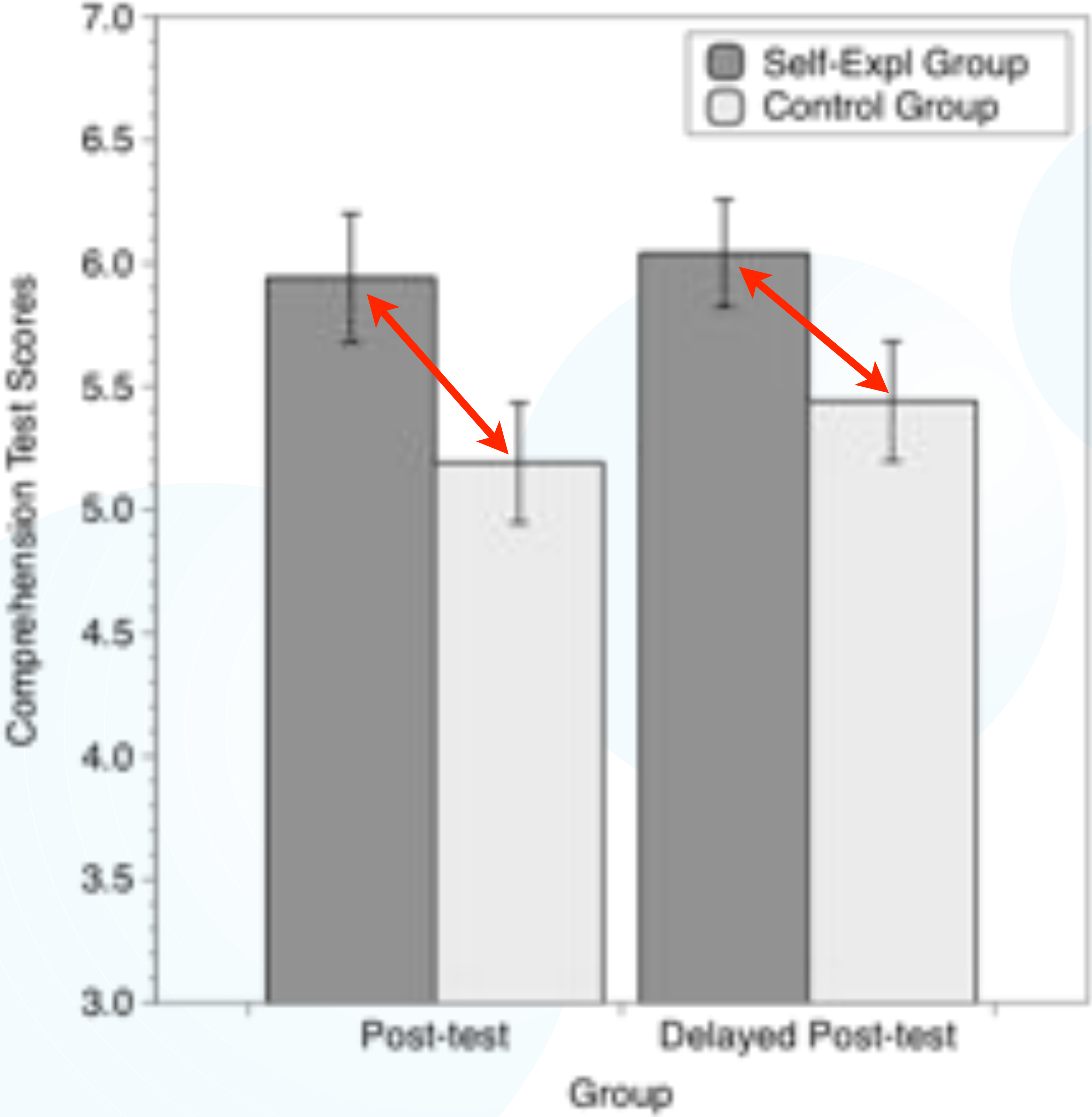


Proof comprehension



Measure: proof comprehension scores (out of 10).

Proof comprehension



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Significant effect of condition ($p = .016$)

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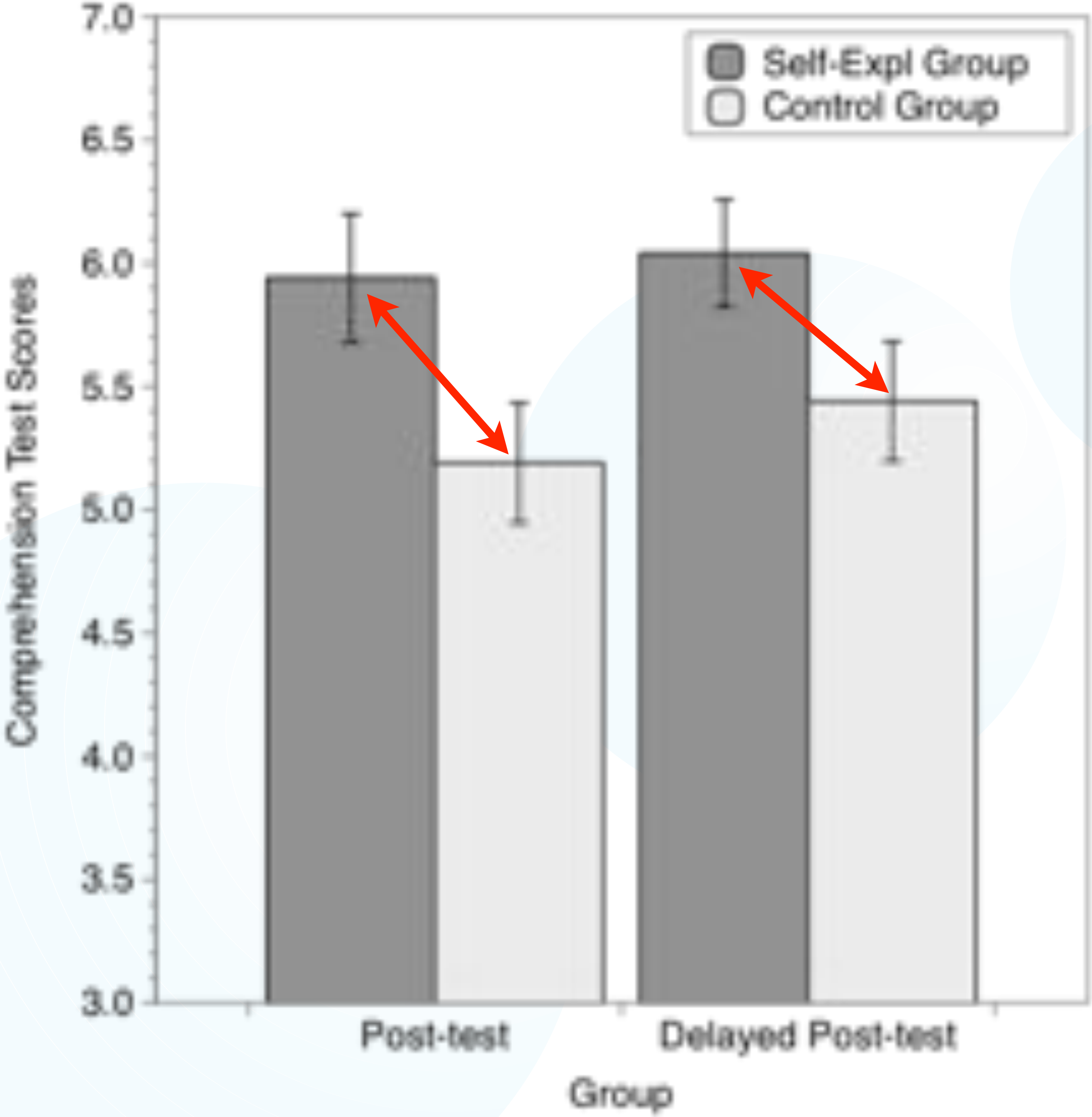

Measure: proof comprehension scores (out of 10).

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No significant effect of time; no significant interaction

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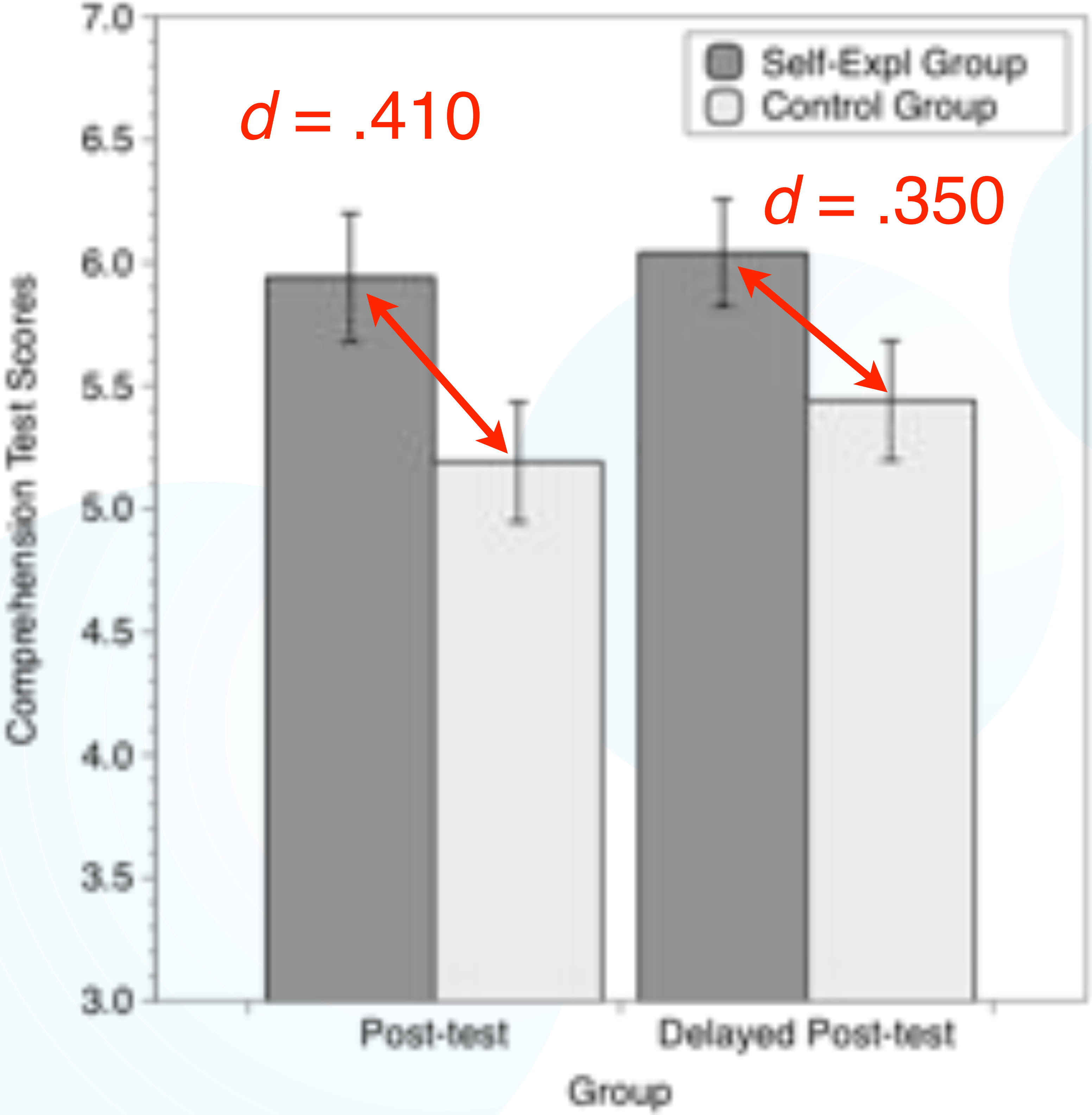

Measure: proof comprehension scores (out of 10).

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Study 3: Summary

- Promising indications that *self-explanation training* may provide a solution to the problem of inefficient reading strategies.
- Needs testing with other types of mathematical texts (i.e. content unrelated to infinity, textbook explanations rather than just proofs).

Overall Summary

- Students find cardinality a difficult topic to engage with (very poor performance on our comprehension test);

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- We found no evidence that this was related to conflicts with intuitions of numbers;
- Rather, problem was considerably more fundamental: students have highly inefficient study strategies which prevents them from engaging with these new mathematical ideas;
- Self-explanation training seems a promising approach.

Thank you

Collaborators:

Lara Alcock
Christian Greiffenhagen
Mark Hodds

Funder:

THE ROYAL SOCIETY
CELEBRATING 350 YEARS

m.j.inglis@lboro.ac.uk

homepages.lboro.ac.uk/~mamji