Finitism and Open-Texture

Luca Incurvati

Magdalene College, Cambridge

History and Philosophy of Infinity
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Concepts

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- Roughly speaking, a criterion of application for a concept `C` tells us to which objects `C` applies.
- Certain concepts are thought to be also associated with a *criterion of identity*.
- A criterion of identity specifies the conditions under which some thing x falling under a concept `C` is the same as another thing y, also falling under `C`.

Ranges of application and disapplication

- Call the *range of application* of a concept the class of objects to which a given concept applies.
- And call the *range of disapplication* of a concept the class of things to which the concept *disapplies*—where 'disapplies' is used as an antonym of 'applies'.

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- And call the *range of disapplication* of a concept the class of things to which the concept *disapplies*—where ‘disapplies’ is used as an antonym of ‘applies’.
- A criterion of application and a criterion of identity for a concept need not settle all questions concerning, respectively, whether the concept applies or disapplies to a certain thing and the identity between objects falling under that concept.

Smidget

- Soames (1999) considers the concept $[x\colon x \text{ is a smidget}]$, associated with the following criterion of application:

(1) 'smidget' applies to x if x is greater than four feet tall

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- Clearly, $[x\colon x \text{ is a smidget}]$ is such that its range of application and its range of disapplication do not exhaust all possibilities: if x is three feet tall 'smidget' neither applies nor disapplies to it.

Waismann

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  Suppose I have to verify a statement such as 'There is a cat next door'; suppose I go over to the next room, open the door, look into it and actually see a cat. Is this enough to prove my statement? . . . What . . . should I say when the creature later on grew to a gigantic size? Or if it showed some queer behaviour usually not to be found with cats, say, if, under certain conditions it could be revived from death whereas normal cats could not? Shall I, in such a case, say that a new species has come into being? Or that it was a cat with extraordinary properties? (Waismann 1945: 121–122)

Open-texture

- Waismann offers the following diagnosis of the situation:

  > *The fact that in many cases there is no such thing as a conclusive verification is connected to the fact that most of our empirical concepts are not delimited in all possible directions. (Waismann 1945: 122)*

- Most of our empirical concepts, Waismann says, display *open-texture*.

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  > *The fact that in many cases there is no such thing as a conclusive verification is connected to the fact that most of our empirical concepts are not delimited in all possible directions. (Waismann 1945: 122)*

- Most of our empirical concepts, Waismann says, display *open-texture*.
- Translated in the terminology we are adopting to say that a concept displays open-texture seems to amount to saying that the range of application and the range of disapplication of a concept do not exhaust all possibilities.

Shapiro on open-texture

- Waismann's focus is on *empirical* concepts; Stewart Shapiro has argued that what Waismann says is true for at least one mathematical concept, namely the concept of *computability*:

> *in the thirties, and probably for some time afterward, [the pre-theoretic notion of computability] was subject to open-texture. The concept was not delineated with enough precision to decide every possible consideration concerning tools and limitations. (Shapiro 2006: 441)*

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- But, Shapiro continues, the mathematical and conceptual work carried out by Turing and the subsequent efforts by the founding fathers of computability theory served to sharpen [$x\colon x$ is computable] into what is now known as the concept of *effective* computability.

Concepts and conceptions

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Mary

Mary is going to be rewarded by her company for her successful work on a certain case. Jane and Susan, however, disagree over whether this reward is fair: Jane thinks that Mary's work has, in fact, entirely been carried out by Mary's colleague Marianne, whilst Susan is persuaded that it has not.

Jill

Like Mary, Jill is going to be rewarded by her company for her work on a certain case. Jane and Susan, however, disagree over whether this decision is fair: according to Jane, Mary should be rewarded because it is fair to reward employees depending on their contribution, whereas according to Susan it is not. For Susan, a company should reward its employees depending on their efforts in their work, regardless of its outcome.

Concepts and conceptions

- In the Jill example, the disagreement concerns the criterion of application for ‘fair’.
- But now recall that it seems reasonable to assume that both Jane and Susan *possess* the concept of fairness: we can say that although both Jane and Susan have the concept of fairness, they have different *conceptions* of it.

Concepts and conceptions

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Conception

A *conception* of `C`, where `C` is a concept, is a (possibly partial) answer to the question ‘What is it to be something falling under `C`?’ which someone could disagree with without being reasonably deemed not to possess `C`.

- N.B.: the distinction need not be *sharp* and need not be *fixed*.

Back to Shapiro

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- Shapiro's suggestion accords with our account of the concept/conception distinction.
- The concept of computability—the *pre-theoretic notion*, as Shapiro puts it—displayed open-texture.
- In order to put the concept to mathematical use, the concept needed to be sharpened by putting forward a *conception* of computability, and various candidates presented themselves: computability as effective computability, but also, for instance, computability as *practicable* computability.
- Eventually, the mathematical community settled for computability as effective computability, on the basis of, *inter alia* considerations about the interest and fruitfulness of this notion.

Hilbert and intuition

- In a much quoted passage, Hilbert writes:

  As a precondition for the application of logical inference and for the activation of logical operations, something must already be given in representation: certain extra-logical discrete objects, which exist intuitively as immediate experience before all thought. If logical inference is to be certain, then these objects must be capable of being surveyed in all their parts, and their presentation, their difference, their succession (like the objects themselves) must exist for us immediately, intuitively, as something which cannot be reduced to something else. (Hilbert 1922: 202. Repeated almost verbatim in Hilbert 1926: 192)

Parsons and intuitive knowledge

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- On this view, finitary mathematics is that part of mathematics that can be known intuitively.
- Parsons' idea is that in perceiving a token inscription of some strokes, such as

  $$|||$$

  we intuit its type, because we see the token *as* a string, and that on this kind of intuition we can base some non-trivial mathematical knowledge.

Parsons on intuition

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- For something to count as intuition, it has to be significantly analogous to perception in various respects: it has to be *de re* and must not involve reflection.
- Why not say that, when presented with a token of a string of strokes, someone who possesses the appropriate conceptual abilities can come to believe that tokens can reliably indicate features of types?
- Probably, Parsons would say that this would not explain the immediacy and obviousness of the relevant knowledge.

Types and vagueness

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- Why not say that numbers, roughly speaking, should be identified with tokens of strings?
- The problem with this proposal is that it introduces within arithmetic features which are foreign to it and which pertain to physical objects.
- Suggestion: overcome this problem by providing a nominalistic reconstruction of types as equivalence classes under the relation *being of the same type*.
- Problem: the distinctive vagueness of physical objects will still imply that it is possible to construe a sorites series of sign tokens for the predicate 'being of the same type' such that the first member of the series is

$$|||$$

and the last member is

$$||\qquad|.$$

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- In other words, even if we find it vague whether

$$\textit{the type instantiated by } ||| = \textit{the type instantiated by } || \qquad |,$$

  we do not need to take this as showing that types themselves are vague.
- We might well take it as showing that it is vague whether a certain token is an instance of a certain type or of another type.
- Once the string has been intuited, Parsons claims, there can be no vagueness about how many strokes it has.

Types and vagueness

What makes intuition mathematical intuition is that it gives objects that instantiate concepts that have a sharp, precise character. At least for statements in the mathematical vocabulary, there is no vagueness in their application to string of strokes. There may indeed be vagueness as to whether what is before us is or is not a token of a given string, but not about the question whether one string, say, consists of two more strokes than another. (Parsons 1993: 237)

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- Note that Parsons' argument, if successful, only defuses a possible objection to his account of mathematical intuition; it does not show that vagueness does not affect types even when it affects their tokens.

PRA and intuitive knowledge

- The thesis that a proof of p in PRA yields intuitive knowledge of p follows, according to Parsons, from the following sub-theses:

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(1) Successor can be seen intuitively to be well-defined.

(2) The elementary successor axioms can be known intuitively.

(3) Logical inference preserves intuitive evidence.

(4) Inference by induction preserves intuitive evidence.

(5) If we introduce a function f by primitive recursion, and the assumed functions needed for f have been intuitively seen to be well-defined, then, on receipt of the recursion equations for f, we can know intuitively that the new function f is well defined.

PRA and intuitive knowledge

- Parsons: (1), (2), (3), (4) hold, but (5) does not.
- Already a weakening of Hilbert's position, since Hilbert took finitary reasoning to encompass at least PRA.

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- Parsons: (1), (2), (3), (4) hold, but (5) does not.
- Already a weakening of Hilbert's position, since Hilbert took finitary reasoning to encompass at least PRA.
- (1) amounts to the thesis that we can know intuitively that every string can be intuited.
- (1) is already problematic, and intuition might take us less far than Parsons supposes.

Why (1)?

- PRA is a quantifier-free theory, and here we do not have the axiom which is naturally interpreted as stating that successor is well-defined:

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Every number has a successor which is also a number.

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(*)

Every number has a successor which is also a number.

- The string analogue of (*) is

(**)

Every stroke string has a successor which is also a stroke string.

- But Parsons makes use of (1) in his argument for (4).

Parsons' argument

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- Parsons needs to show how the kind of general knowledge involved in knowledge of (**) is intuitive in the sense of being based on intuition of objects.He writes:

  One has to imagine an arbitrary string of strokes [. . .] There seems to be a choice between imagining [a stroke string inscription] vaguely, that is imagining a string of strokes without imagining its internal structure clearly enough so that one is imagining a string of n strokes for some particular n, or taking as paradigm a string (which now might be perceived rather than imaged) of a particular number of strokes, in which case one must be able to see the irrelevance of this internal structure, so that in fact it plays the same role as the vague imagining. (Parsons 1980: 156–157)

Two strategies

- Parsons' two strategies resemble the two strategies pursued by Locke and Berkeley to deal with the problem of generality
- In the first case, we can know something about all strings of strokes by knowing something about a general stroke.
- In the second case, we can know something about all strings of strokes by knowing something about a particular string of strokes and by seeing that there is nothing particular to that string that prevents the result from extending to all strings.

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- The two strategies are reflected in the contemporary debate on how to read the rule of universal generalization.

The first strategy

- Page (1993): if what we are presented with is a vaguely imagined stroke string, we cannot be sure that the initial string consisted of n strokes and that the result of adding a new stroke is a string of $n+1$ strokes.
- Parsons' (1993) reply: Page is confusing (**) with the claim that $Sx \neq x$, and Parsons agrees that we cannot establish the latter by intuition.

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- Parsons' (1993) reply: Page is confusing (**) with the claim that $Sx \neq x$, and Parsons agrees that we cannot establish the latter by intuition.
- Galloway (1999): in formal arithmetic we can tell whether the string obtained by adding a stroke to the original one is not of the same length of the original only by induction, and it is difficult to see any other way to establish the result which does not involve reflection.
- Reply on behalf of Parsons: to be said to know intuitively that the successor operation is well-defined we cannot be required to know intuitively that $Sx \neq x$, since this can be proved only by induction, which, he thinks, cannot be intuitively known.

The first strategy

- However, in general we distinguish between (**) and $Sx \neq x$ because there can be a deviant interpretation under which the domain comprises, in addition to the natural numbers, non-standard elements.
- In the situation we are considering, on the other hand, the impossibility of ruling out that $Sx = x$ is due to the vagueness of the imagined string.

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- In the situation we are considering, on the other hand, the impossibility of ruling out that $Sx = x$ is due to the vagueness of the imagined string.
- Parsons talks of imagining vaguely a string of strokes.
- Plausible to distinguish between imagining as conceiving and imagining as having a mental image.

Two versions of the first strategy

- First option: imagining the string of strokes seems to be nothing but conceiving of the string as constructed step by step.
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- Second option: form the mental image of a vague string. But what will this image look like?
- First possibility: we form the image of a vague string in the sense of a general string, which presumably will look something like

$$|\ldots|$$

  We are then supposed to add a stroke to it in imagination, obtaining something like:

$$|\ldots||$$

  The result, then, applies to all strings.
- But this assumes that the '...' is given in intuition, which was what to be shown.

Two versions of the first strategy

- Second possibility: we form the mental image of a vague string, in the sense of a mental image of a string whose internal structure is not clear enough to determine whether it is composed of n strokes for some particular n.
- Difficult to represent such a string on a piece of paper, but even if we grant that such a string can indeed be imagined, it is hard to see whether the fact that a stroke can be added to it shows anything more than some string or other can be extended.

Two versions of the first strategy

- Second possibility: we form the mental image of a vague string, in the sense of a mental image of a string whose internal structure is not clear enough to determine whether it is composed of n strokes for some particular n.
- Difficult to represent such a string on a piece of paper, but even if we grant that such a string can indeed be imagined, it is hard to see whether the fact that a stroke can be added to it shows anything more than some string or other can be extended.
- Moreover, recall that in order to avoid the unpalatable consequences of introducing questions of individuation within the realm of arithmetic, Parsons has claimed that there can only be vagueness as to whether a given token is a token of a given string.
- But then, it seems that when we imagine a vague string we are either not having an intuition after all or just being in doubt as to whether the imagined string is a token of a given string or not.

The second strategy

- First, imagine or perceive a particular string and see that it is extendible.
- Then, see that there is nothing relevant in its internal structure to its being extendible, so that the result applies to *any* string.
- It is difficult to see, however, how this argument can show that (**) can be known *intuitively*, since the argument involves seeing that the internal structure of the string is irrelevant to its extendibility, and this seems to require reflection.

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- It is difficult to see, however, how this argument can show that (**) can be known *intuitively*, since the argument involves seeing that the internal structure of the string is irrelevant to its extendibility, and this seems to require reflection.
- To appreciate this, notice that there does not seem to be any difference, as far as intuition is concerned, between seeing the passage from ||| to |||| as delivering knowledge about the possibility of passing from a particular string to another particular string and seeing it as delivering general knowledge about the passage from a string of n strokes for any n to a string of $n+1$ strokes.

Tait

- Tait (1981: 524): 'the "finite" in "finitism" means precisely that all reference to infinite totalities should be rejected'.

Tait's Thesis

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Tait's Thesis

The finitistic functions are precisely the primitive recursive functions.

- Tait: finitistic maths coincides with PRA plus the universal closures of the numerical equations provable in it.
- A related proposal is that PRA is sound and complete for finitistic maths (sometimes also called 'Tait's Thesis').

Infinite totalities

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Infinite totalities

- Tait sometimes talks of *assumptions* of infinity, while he characterizes finitism as rejection of *reference* to infinite totalities.
- Distinguish between the following ways in which a theory T can appeal to the existence of infinite totalities:
  - (a) T proves or recognizes as true a statement which says that there is an infinite object.
  - (b) The variables of T can only be interpreted as ranging over an infinite domain.
  - (c) Understanding T involves infinitary concepts.

On (a) and (b)

- When one talks of reference to infinite totalities, one usually has in mind (a).
- Cannot be what Tait has in mind: Peano Arithmetic proves many statements such as $\forall x \forall y(x + y = y + x)$, but does not prove statements which say that there is an infinite object.

On (a) and (b)

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- Cannot be what Tait has in mind: Peano Arithmetic proves many statements such as $\forall x \forall y(x + y = y + x)$, but does not prove statements which say that there is an infinite object.
- (b) seems to be what Tait has in mind:
  - (i) explains Tait's assertion that statements such as $\forall x \forall y(x + y = y + x)$ seem 'to refer to the infinite totality of numbers', since talking loosely, a quantified sentence can be said to refer to the domain over which its quantifiers range, in the sense that it talks about them.
  - (ii) since PRA is a quantifier-free theory, it is susceptible of an interpretation which makes it finitistic in the sense (b).

PRA and (b)

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- In PRA we can prove numerical equations with free-variables.
- In logic textbooks, these free-variable formulae are treated semantically as if they were universally quantified.
- However, the quantifier-free formulae of PRA can be regarded as recording the fact that it has been shown finitistically that $F(a)$ is a correct numerical formula for each a.
- When interpreted this way, quantifier-free formulae do not seem to involve any commitment to an infinite totality in the sense (b).

PRA and (b)

- This reading of general propositions as schemata cannot be extended to the universal statements that can be proved in Peano Arithmetic.
- The reason is, crucially, that universal statements in PA are capable of negation.
- Moreover, the possibility of negating universal statements in PA allows us to apply the law of excluded middle to them, so that we can prove statements of the form $\forall x F(x) \vee \exists x \neg F(x)$.

PRA and (b)

- This reading of general propositions as schemata cannot be extended to the universal statements that can be proved in Peano Arithmetic.
- The reason is, crucially, that universal statements in PA are capable of negation.
- Moreover, the possibility of negating universal statements in PA allows us to apply the law of excluded middle to them, so that we can prove statements of the form $\forall x F(x) \vee \exists x \neg F(x)$.
- Hilbert often formulates the law of excluded middle as $\neg \forall x F(x) \rightarrow \exists x \neg F(x)$, which makes it look as if he thinks that the classical laws of logic are in need of justification only when one applies them to general statements in the way we are allowed to do in PA.
- There is a technical result that seems to confirm that Hilbert was right about this: as Curry and Goodstein independently showed, PRA can be formulated as a logic-free calculus based solely on equations.

On (c)

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- It is sometimes said that PA_2 is infinitary in sense (c).
- The claim is that to understand the notion of arbitrary subsets of natural numbers we need to be able to make sense of the idea of a completed totality of natural numbers.
- Whether or not this is correct, first-order PA does not seem to be infinitary in this sense either.
- Hence, Tait's arguments for distinguishing between Primitive Recursive Arithmetic and Peano Arithmetic on the basis of the distinction between finite and infinite depend on a certain way of spelling out 'appeal to infinite totalities'.

Cartesian finitism

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Cartesian finitism

- Tait rejects Hilbert's characterization of finitary mathematics in terms of intuition.
- Instead, Tait characterizes finitistic reasoning as that part of mathematical reasoning which cannot be doubted without giving up mathematical reasoning altogether:

  [A]lthough we cannot speak of the absolute security of finitism, there is a sense in which we can speak of its indubitability. That is, any nontrivial reasoning about numbers will presuppose finitist methods, and there can be no preferred or even equally preferable standpoint from which to launch a critique of finitism. In other words, it is simply pointless to doubt it. (Tait 1981: 546)

Cartesian finitism

- Kind of Cartesian strategy: retreat to that part of mathematics which cannot be coherently doubted and found the rest of our mathematical knowledge on it.
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- Kind of Cartesian strategy: retreat to that part of mathematics which cannot be coherently doubted and found the rest of our mathematical knowledge on it.
- Tait's reconstruction starts from the generality and indispensability for thought of finitistic mathematics.
- Tait thinks that PRA is the theory in which any commitment to infinite totalities is avoided.
- However, he also thinks that PRA is that part of mathematics indubitable in a Cartesian sense.

Cartesian finitism and iteration

- Worry: systems weaker than PRA which allow non-trivial results and seem therefore more suitable to represent minimal numerical reasoning.

Cartesian finitism and iteration

- Worry: systems weaker than PRA which allow non-trivial results and seem therefore more suitable to represent minimal numerical reasoning.
- Example: *Elementary Arithmetic* (EA), in which one has the quantifier-free axioms for successor, identity, addition, multiplication and exponentiation and the schema of induction for bounded formulae.
- This system is weaker than PRA (class of functions provably recursive in EA is the class of Kalmar elementary functions).
- Harvey Friedman, in various posts on FOM has conjectured that every arithmetical theorem published in the Annals of Mathematics can be proved in EA.

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  *[*PRA*] is characterized by two moments. One is that the basic principle of definition of functions, aside from explicit definition, is definition by iteration or primitive recursion. [...] The other moment of* PRA *is that we apply iteration or primitive recursion only to operations on domains D of finite objects—call them finitary domains. [...] These two moments determine* PRA, *and this explains its special 'minimal' role and why it deserves the title 'finitary' (independently of any historical uses of this term): it contains the principle that defines our concept of number, but applies it only to operations on finitary domains. (Tait 2006: 86)*

Cartesian finitism and iteration

- The idea is therefore that iteration is part of the concept of number and that there can be no genuine numerical reasoning without it.
- Since the idea of iteration is enough to justify the class of functions closed under primitive recursion, the notion of iteration is enough to justify PRA, unless we restrict it arbitrarily.

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- The idea is therefore that iteration is part of the concept of number and that there can be no genuine numerical reasoning without it.
- Since the idea of iteration is enough to justify the class of functions closed under primitive recursion, the notion of iteration is enough to justify PRA, unless we restrict it arbitrarily.
- This suggestion, however, raises a number of problems, even if we leave aside the question whether the restrictions on iteration imposed in subsystems of PRA are less legitimate that the one imposed in PRA itself.
- For, even granted that the concept of iteration is part of our ordinary concept of number, it still has to be shown that there is no genuine numerical reasoning without it.

Counting and iteration

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We discern finite sequences in our experience—sequences of words on a page, of peals of a bell, of people in a room ordered by age or size or simply by counting. We not only discern such sequences but we see them as sequences, i.e. as having the form of finite sequences. I shall call this form Number. [...] We understand the numbers as the specific determinations of Number. For example we see

$$1111111111111111111$$

as a finite sequence. Then we can count and determine its number. If it were otherwise, what would we be counting? (Tait 1981: 529)

Counting and iteration

A finite sequence is obtained from the null sequence by iterating the operation of taking one-element extensions. Thus a number is obtained from 0 by iterating the operation $m \mapsto S(m)$ *of taking successors. [...] This is a basic fact, constitutive of our idea of finite sequence. (Tait 1981: 531–532)*

Counting and iteration

> *A finite sequence is obtained from the null sequence by iterating the operation of taking one-element extensions. Thus a number is obtained from 0 by iterating the operation $m \mapsto S(m)$ of taking successors. [...] This is a basic fact, constitutive of our idea of finite sequence. (Tait 1981: 531–532)*

(1) One can count how many *Fs* there are only if one can see the *Fs* as a finite sequence.
(2) One can see the *Fs* as a G only if one has the idea of a G.
(3) One can see the *Fs* as a finite sequence only if one has the idea of a finite sequence.
(4) The idea of a finite sequence presupposes that it is built up from a first element by iterating the successor operation.
(5) The idea of a finite sequence presupposes the idea of iteration.
(6) One can count how many *Fs* there are only if one has the idea of iteration.

Why the argument fails

- The argument is based on a fallacy of equivocation.
- If we take 'sequence' as denoting a minimal notion of sequence, necessary to count how many *Fs* there are, then premise (1) is true, but premise (4) is false since to grasp this notion of sequence one need not have the idea of a sequence as something which is built up from a first element by using the successor operation.

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- If, on the other hand, we take 'sequence' as standing for a richer notion of sequence, something which is built up from a first element by using the successor operation, then premise (4) is obviously true, but premise (1) is false, because it is possible to count how many *Fs* there are without seeing the *Fs* as such a kind of sequence.

Back to conceptions and open-texture

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Back to conceptions and open-texture

- There seem to be, therefore, at least two different broad conceptions of Hilbertian finitism.
- On the first conception, finitistic mathematics is that part of mathematics which can be considered as finite in some interesting sense.
- On the second conception, finitistic mathematics is that part of mathematics which enjoys some particular epistemological status due to its intuitive knowability or to its indubitability.

Gödel

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Gödel

- These two conceptions, far from being co-extensive, seem to pull in opposite directions.
- As far as intuitive knowability is concerned this conclusion is entirely in agreement with a remark made by Gödel:

Note that it is Hilbert's insistence on concrete knowledge that makes finitary mathematics so surprisingly weak [. . .] There is nothing in the term 'finitary' which would suggest a restriction to concrete knowledge. Only Hilbert's special interpretation of it makes this restriction. (Godel 1972: 272)

Hilbert & Bernays

- In the second volume of the *Grundlagen der Mathematik*, Hilbert and Bernays write:

  We have introduced the expression 'finitistic' not as a sharply delimited term, but only as the name of a methodological guideline, which enables us to recognize certain kinds of concept-formations and ways of reasoning as definitely finitistic and others as definitely not finitistic. This guideline, however, does not provide us with a precise demarcation between those which accord to the requirements of the finitistic method and those that do not. (H&B 1939: 347–348)

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- This accords with thinking of finitism as a concept displaying open-texture.

Hilbert & Bernays and open-texture

- Presumably, on any conception of finitism the basic arithmetic of elementary numerical equations will turn out to be finitistic and the modes of reasoning employed in ZF will result as non-finitistic.

Hilbert & Bernays and open-texture

- Presumably, on any conception of finitism the basic arithmetic of elementary numerical equations will turn out to be finitistic and the modes of reasoning employed in ZF will result as non-finitistic.
- In the middle, there are a number of theories which count as finitistic depending on which conception of finitism we choose:
  - epistemological, based on intuitive knowability or indubitability;
  - or semantical and ontological, based on the distinction between the finite and the infinite, which in turn can be modelled in terms of commitment to infinite totality, reference to infinite totality, or use of infinitary concepts.

Finitism and Open-Texture

Luca Incurvati

Magdalene College, Cambridge

History and Philosophy of Infinity
Corpus Christi, Cambridge
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