How to measure the size of sets: Aristotle-Euclid or Cantor-Zermelo?

Marco Forti

Dipart. di Matematica - Università di Pisa

forti@dma.unipi.it

Joint research with

Vieri Benci and Mauro Di Nasso

Conference FotFS VIII:

"History and Philosophy of Infinity"

Corpus Christy - Cambridge, September 20^{th}-23^{rd}, 2013.

A measure of size for arbitrary sets should be submitted to the famous five common notions of Euclids Elements, which traditionally embody the properties of magnitudes,

1. *Things equal to the same thing are also equal to one another.*
2. *And if equals be added to equals, the wholes are equal.*
3. *And if equals be subtracted from equals, the remainders are equal.*
4. *Things applying* [exactly] *onto one another are equal to one another.*
5. *The whole is greater than the part.*

The presence of the fourth and fifth principles among the Common Notions in the original Euclid's treatise is controversial, notwithstanding the fact that they are explicitly accepted in the fundamental commentary by Proclus to Euclid's Book I, where all the remaining statements included as axioms by Pappus and others are rejected as spurious additions. We consider the five principles on a par, since all of them can be viewed as basic assumptions for any reasonable theory of magnitudes.

NB We translate $\epsilon\phi\alpha\rho\mu o\zeta o\nu\tau\alpha$ by *"applying* [exactly] *onto"*, instead of the usual *"coinciding with"*. This translation seems to give a more appropriate rendering of the Euclidean usage of the verb $\epsilon\phi\alpha\rho\mu o\zeta\epsilon\iota\nu$, which refers to superposition of congruent figures.

This remark is important, because in measuring infinite collections it has to be taken much weaker than the full Cantorian counting principle that equipotent sets have equal sizes.

The notion of numerosity

A notion of "number of elements" (numerosity) that maintains the Euclidean principle that the whole is larger than the part for infinite collections was first introduced in [1] for so called "labelled sets", a special class of countable sets whose elements come with natural numbers as labels.

This notion of numerosity was then variously generalized in several papers:

- to arbitrary sets of ordinal numbers in [3],
- to whole "universes of mathematical objects" in [4], and, returning to the original Cantorian study,
- to finite dimensional real point sets in [5], while
- special numerosities of point sets over a countable line are considered in [6,7].

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Cantor's theory of cardinalities

The Cantorian theory of cardinalities originates from the idea of extending the notion of counting from finite sets to arbitrary Punktmengen, *i.e.* arbitrary subsets of the Euclidean space $\mathbb{E}_d(R)$ of dimension d. Cantor assumed the natural counting principle

(CP): two point sets have the same cardinality if and only if they can be put in one-to-one correspondance.

Incompatibility of Cantor with Euclid

Historically the Cantorian principle (CP) revealed incompatible with the fifth Euclidean common notion, for infinite collections, long before the celebrated Galileo's remark that there should be simultaneously "equally many" and "much less" perfect squares than natural numbers.

The impact of this inconsistency cannot be overestimated: it led Leibniz (an inventor of infinitesimal [nonstandard] analysis!) to assert the *impossibility of infinite numbers*.

Cantor extended sum, product, and ordering of integers to infinite cardinals by assuming the following natural principles:

(SP): $|A| + |B| = |A \cup B|$ provided $A \cap B = \emptyset$.

(PP): $|A||B| = |A \times B|$ for all A, B.

(OP): $|A| \leq |B|$ if and only if there exists $C \subseteq B$ such that $|A| = |C|$.

The algebra of cardinalities

In the Cantorian theory of cardinalities these properties give rise to a very weird arithmetic, where

$$\mathfrak{a} + \mathfrak{b} = \max(\mathfrak{a}, \mathfrak{b})$$

whenever the latter is infinite.

No cancellation law (*a fortiori* no subtraction), hence not only the 5^{th}, but also the 3^{rd} Euclidean principle fails. (*i.e.* Aristotle's preferred example of an axiom.)

Actually, every infinite set is equipotent to a proper subset (*Dedekind's negative definition of finiteness)*

Moreover, not allowing division, Cantor's theory provides a satisfying treatment of infinitely large numbers, but it cannot produce "infinitely small" numbers, thus preventing a natural introduction of "infinitesimal analysis".

(History repeats itself: Cantor asserted the existence of actually infinite numbers, but strongly negated that of actually infinitesimal numbers!)

The 1^{st} Euclidean principle for collections

- *Things equal to the same thing are also equal to one another*

essentially states that "having equal sizes" is an equivalence. We write $A \approx B$ when A and B are equinumerous (have equal sizes). The first Euclidean principle becomes

E1 (Equinumerosity Principle)

$$A \approx C,\ B \approx C \implies A \approx B.$$

2^{nd} and 3^{rd} Euclidean principles for collections

- *And if equals be added to equals, the wholes are equal*
- *And if equals be subtracted from equals, the remainders are equal*

addition and subtraction are “compatible” with equinumerosity. For collections, sum and difference naturally correspond to disjoint union and relative complement, *à la* Cantor:

E2 (Sum Principle)

$$A \approx A', B \approx B', A \cap B = A' \cap B' = \emptyset \Longrightarrow A \cup B \approx A' \cup B'$$

E3 (Difference Principle)

$$A \approx A',\ B \approx B',\ B \subseteq A,\ B' \subseteq A' \ \Longrightarrow A \setminus B \approx A' \setminus B'$$

The 5^{th} Euclidean principle for collections

- *The whole is greater than the part*

Say that A is smaller than B, written $A \prec B$, when A is equinumerous to a proper subset of B

$$A \prec B \iff A \approx A' \subset B$$

Comparison of sizes must be consistent with equinumerosity. So the fifth principle becomes

E5 (Ordering Principle)

$$A \subset B \approx B' \implies A \not\approx B \ \& \ A \prec B'$$

The problem of comparability

Homogeneous magnitudes are usually arranged in a linear ordering. So the followig strengthening of the Ordering Principle would be most wanted

E5b (Total Ordering Principle)

Exactly one of the following relations always holds:

$$A \prec B, \quad A \approx B, \quad B \prec A$$

[A weaker alternative could be requiring **E5b** only for a transitive extension of the relation $\prec$]

• Cardinalities of infinite sets are always comparable, but only thanks to Zermelo's Axiom of Choice.

The algebra of numerosities

Measuring size amounts to associating suitable "numbers" (numerosities) to the equivalence classes of equinumerous collections. Sum and ordering of numerosities can be naturally defined *à la Cantor*

(sum) $\mathfrak{n}(X)+\mathfrak{n}(Y)=\mathfrak{n}(X\cup Y)$ whenever $X\cap Y=\emptyset$;

(ord) $\mathfrak{n}(X)\leq\mathfrak{n}(Y)$ if and only if $X\preceq Y$.

thanks to the principles **E2**, **E3**, and **E5a**.

A "satisfactory" algebra of numerosities should also comprehend a product, so as to obtain (the non-negative part of) a (discretely) ordered ring.

The product of numerosities

One could view the notion of measure as originating from the length of lines, and later extended to higher dimensions by means of products. In classical geometry, a product of lines is usually intended as the corresponding rectangle. So one could use Cartesian products in defining the product of numerosities.

The natural "arithmetical" idea that multiplication is an iterated addition of equals is consistent with the "geometrical" idea of rectangles, because the Cartesian product $A \times B$ can be naturally viewed as the union of "B-many disjoint copies" of A

$$A \times B = \bigcup_{b \in B} A_b, \text{ where } A_b = \{(a,b) \mid a \in A\}.$$

The Product Principle

But is A_b a "faithful copy" of A?

- Let $A = \{b, (b,b), ((b,b),b), \ldots, (((\ldots,b),b),b), \ldots\}$

$A_b = A \times \{b\}$ is a proper subset of A, so (the numerosity of) the singleton $\{b\}$ is not an identity w.r.t. (the numerosity of) A.

A severe constraint, stronger than disjointness condition of the Sum Principle, has to be put in the following

PP (Product Principle)

$$A \approx A',\ B \approx B' \implies A \times B \approx A' \times B'$$

e.g. considering only finite dimensional point sets, *i.e.* subsets of the n-dimensional spaces $\mathbb{E}_n(\mathcal{L})$ over arbitrary "lines" $\mathcal{L}$.

The 4^{th} Euclidean principle for collections

- *Things applying* [exactly] *onto one another are equal to one another*

> ... the [fourth] Common Notion ... is intended to assert that superposition is a legitimate way of proving the equality of two figures ... or ... to serve as an *axiom of congruence*. (T.L. Heath).

i.e. "appropriately faithful" transformations (*congruences*) preserve sizes: it is a criterion for being equinumerous.

Isometries vs. congruences

The 3^{rd} (and 5^{th}) common notion can be saved by restricting the meaning of *"applying* [exactly] *onto"* to comprehend only "natural transformations", such as permutations and regroupings of n-tuples, and similars. Any transformation T with an infinite orbit $\Gamma = \{x, Tx, T^2x, \dots\}$ maps Γ onto a proper subset of Γ, so T is not a "congruence" for Γ itself.

Also the isometries of Euclidean geometry work only for *special classes of bounded geometrical figures*. Even on the line there are no translation invariant numerosities [MF 2010]:

- There exist bounded subsets of the algebraic line $\bar{\mathbb{Q}}$ that are proper subsets of some of their translates

Natural congruences

A notion of congruence appropriate for the 4^{th} Euclidean principle might include all “natural transformations” that map tuples to tuples having the same sets of components

- A natural congruence is an injective function mapping tuples to tuples with the same components.

E4a (Natural Congruence Principle)

$X \approx T[X]$ for all natural congruences T.

A wider class of “natural congruences”, like rising dimension, is admissible only after “appropriately restricting” their domains of application to finite dimensional point sets.

Restricted isometries

An interesting point of view considers equinumerosity as witnessed by an appropriate family of "restricted isometries":

IP[$\mathcal{T}$] (Isometry Principle) *There exists a group of transformations* $\mathcal{T}$ *such that*

$$A \approx B \quad \Longleftrightarrow \quad \exists T \in \mathcal{T} \quad A \subseteq \mathrm{dom}\, T \ \&\ B = T[A].$$

Remark: IP[$\mathcal{T}$] implies the Half Cantor Principle **HCP** of [3]

$$A \approx B \implies |A| = |B|,$$

and also the ordering principle **E5**

$$A \subset B \approx B' \quad \Longleftrightarrow \quad A \prec B'$$

The algebra of numerosities

A surjective map $\mathfrak{n} : \mathbb{W} \to \mathfrak{N}$ is a numerosity function corresponding to the equinumerosity relation $\approx$ if

$$\mathfrak{n}(X) = \mathfrak{n}(Y) \iff X \approx Y.$$

Define $+, \cdot$ and $<$ on $\mathfrak{N}$ by

(sum) $\mathfrak{n}(X) + \mathfrak{n}(Y) = \mathfrak{n}(X \cup Y)$ whenever $X \cap Y = \emptyset$;

(prod) $\mathfrak{n}(X) \cdot \mathfrak{n}(Y) = \mathfrak{n}(X \times Y)$ for all X, Y;

(ord) $\mathfrak{n}(X) < \mathfrak{n}(Y)$ if and only if $X \prec Y$.

- The numerosity $\mathfrak{n}$ is Zermelian iff $\approx$ satisfies the total ordering principle **E5b**.
- The numerosity $\mathfrak{n}$ is Cantorian iff $\approx$ satisfies the Half Cantor principle **HCP**.

Obviously, finite sets receive their "number of elements" as numerosities, namely

Proposition. *Let A, B be finite. Then*

$$A \approx B \iff |A| = |B|$$

Moreover, if X is infinite, then $A \prec X$.

Theorem. *The structure $\langle \mathfrak{N}, +, \cdot, < \rangle$ is a positive subsemiring of a partially ordered discrete ring, and $\mathbb{N}$ can be taken as an initial segment of $\mathfrak{N}$. $\mathfrak{N}$ is the positive part of a discretely ordered ring if and only if $\approx$ is Zermelian.*

So the algebra of numerosities is the most natural, and in fact, in all cases known up to now, the set $\mathfrak{N}$ can be taken to be a nonstandard extension of the natural numbers.

Set theoretic commitments

• The existence of Cantorian numerosity functions defined for all sets is provable in ZFC.

• Assuming the Continuum Hypothesis there are Zermelian numerosities defined on all countable finite dimensional pointsets over the real line.

• A sufficient condition for the existence of Zermelian numerosities defined on all countable subsets of some cardinal λ is that both $\kappa^{\aleph_0} = \kappa^+$ and $\Box_\kappa$ hold for all singular cardinals $\kappa < \lambda$ of countable cofinality. In particular $V = L$ yields a Zermelian numerosity to all countable sets.

Normal approximations

Let X be uncountable. A normal approximation of X is a sequence $\langle X_\alpha \mid \alpha < |X| \rangle$ s.t.

1. $\alpha < \beta \Longrightarrow X_\alpha \subseteq X_\beta$;
2. $X_\lambda = \cup_{\alpha<\lambda} X_\alpha$ *for limit* $\lambda < |X|$;
3. $|X_\alpha| < |X|$ for all $\alpha < |X|$;
4. $X = \cup_{\alpha<|X|} X_\alpha$.

Normal numerosities

The numerosity function $\mathfrak{n}$ is normal if, given normal approximations $\langle X_\alpha \mid \alpha < |X| \rangle$ of X and $\langle Y_\alpha \mid \alpha < |Y| \rangle$ of Y, one has

$$\forall \alpha < |X| \, (\mathfrak{n}(X_\alpha) \leq \mathfrak{n}(Y_\alpha)\,) \Longrightarrow \; \mathfrak{n}(X) \leq \mathfrak{n}(Y)$$

Let $\mathbb{W}_0$ be the family of all countable point sets over $\mathcal{L}$, and let $\mathbb{W}$ be the family of all point sets over $\mathcal{L}$. Then

Theorem 1. *There exist normal numerosity functions on $\mathbb{W}$ whenever $|\mathcal{L}| < \aleph_\omega$. Moreover, if $\mathfrak{n}$ is a Zermelian numerosity function on $\mathbb{W}_0$, then $\mathfrak{n}$ can be extended to numerosity functions $\mathfrak{n}_n$ and $\mathfrak{n}_C$ on $\mathbb{W}$ that are respectively normal and Cantorian.*

Open questions

- Is it consistent with ZFC the existence of Zermelian numerosities defined on all point sets of an uncountable line?
- Is it possible to have numerosity functions defined on all pointsets that are simultaneously Cantorian and normal?
- Can one prove the existence of normal numerosities defined on all point sets over an arbitrarily large line?

Can Euclid be reconciled with Cantor?

The answer could be

YES! if we recall that the cardinal comparability of all sets was obtained much later by Zermelo,

but

NOT YET! if we take into account that both Euclid and Cantor viewed general comparability as a fundamental principle.