Mathematics in Engineering and Science

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Mathematics in Engineering and Science

- The nature of mathematical reasoning
- Some historical examples
- Computational Science: Emergence of a new reasoning style

Babylonian versus Geek styles in Mathematics

Babylonian style

In Babylonian schools in mathematics the student would **learn something by doing a large number of examples until he caught on to the general rule**. Also he would know a large amount of geometry, a lot of the properties of circles, the theorem of Pythagoras, formulae for the areas of cubes and triangles; in addition, some degree of argument was available to go from one thing to another.

…

Greek or Euclidean style

But Euclid discovered that there was a way in which all the theorems of geometry could be ordered from a set of axioms that were particularly simple. …**The most modern mathematics concentrates on axioms and demonstrations within a very definite framework of conventions of what is acceptable and what is not acceptable as axioms.**

Richard P. Feynman: The Character of Physical Law, Penguin Books 1965, p. 46

The Nature of Mathematics

… mathematics is the science of skillful operations with concepts and rules invented just for this purpose. **The principal emphasis is on the invention of concepts. Mathematics would soon run out of interesting theorems if these had to be formulated in terms of the concepts which already appear in the axioms.**

… the mathematician could formulate only a handful of interesting theorems without defining concepts beyond those contained in the axioms and that the concepts outside those contained in the axioms are defined with **a view of permitting ingenious logical operations which appeal to our aesthetic sense both as operations and also in their results of great generality and simplicity**.
(Eugene Wigner: The Unreasonable Effectiveness of Mathematics in the Natural Sciences.)

The empirical law of epistemology

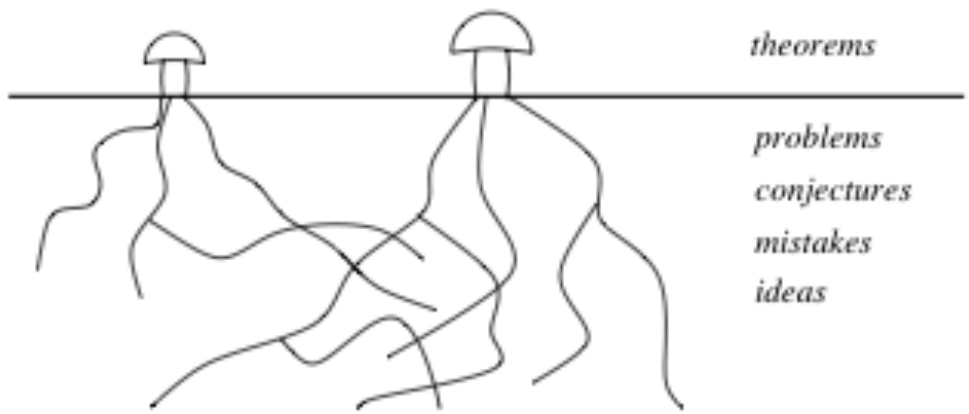
… the "laws of nature" being of almost **fantastic accuracy** but of **strictly limited scope**. I propose to refer to the observation which these examples illustrate as **the empirical law of epistemology**. **Together with the laws of invariance of physical theories, it is an indispensable foundation of these theories**. Without the laws of invariance the physical theories could have been given no foundation of fact; if the empirical law of epistemology were not correct, we would lack the encouragement and reassurance which are emotional necessities, without which the "laws of nature" could not have been successfully explored.

…

Every empirical law has the disquieting quality that one does not know its limitations. We have seen that there are regularities in the events in the world around us which can be formulated in terms of mathematical concepts with an uncanny accuracy. **There are, on the other hand, aspects of the world concerning which we do not believe in the existence of any accurate regularities. We call these initial conditions.** The question which presents itself is whether the different regularities, that is, the various laws of nature which will be discovered, will fuse into a single consistent unit, or at least asymptotically approach such a fusion. Alternatively, it is possible that there always will be some laws of nature which have nothing in common with each other.

(Eugene Wigner: The Unreasonable Effectiveness of Mathematics in the Natural Sciences.)

V.I. Arnold's Mathematical Mushroom



When you are collecting mushrooms, you only see the mushroom itself. But if you are a mycologist, you know that the real mushroom is in the earth. There's an enormous thing down there, and you just see the fruit, the body that you eat. In mathematics, the upper part of the mushroom corresponds to theorems that you see. But you don't see the things which are below, namely ***problems, conjecture, mistakes, ideas, and so on.***

(V.I. Arnold: From Hilbert's Superposition Problem to Dynamical Systems, Mathematical Events of the Twentieth Century, 2006, 19-20.)

“Pure” Mathematics

It should perhaps be stressed again that **the boundaries between mathematics and the many disciplines to which it is applied are seldom sharply drawn. Nothing but impoverishment can be expected from the unfortunately rather frequent current efforts to isolate a body of ‘pure’ mathematics from the rest of scientific endeavour and to let it feed only on itself.**

(Mark Kac and Stanislaw M. Ulam: Mathmatics and Logic, Penguin Books 1968, p. 180)

Garrett Birkhoff

Progress would have been much slower if rigorous mathematics had not been supplemented by various *plausible intuitive hypotheses*. Of these, the following have been especially suggestive:

(A) Intuition suffices for determining which physical variables require consideration.
(B) Small causes produce small effects, and infinitesimal causes produce infinitesimal effects.
(C) Symmetric causes produce effects with the same symmetry.
(D) The flow topology can be guessed by intuition.
(E) The processes of analysis can be used freely: the functions of rational hydrodynamics can be freely integrated, differentiated, and expanded in series (Taylor, Fourier) or integrals (Laplace, Fourier).
(F) Mathematical problems suggested by intuitive physical ideas are "well set".

Garrett Birkhoff: Hydrodynamics: A Study in Logic, Fact, and Similitude. Second Edition 1960, p. 4

Practical Mathematical Reasoning

- **The final mathematical proof must comply with classical (or intuitionistic) logic**
- **Logical validity (consistency) is a necessary part of mathematical truth**
- **Proof constructions usually require introduction of new concepts**
- **Proof constructions are often governed by intuitively defined theorems (hypotheses, conjectures)**
- **New important theorems and hypotheses are often codifications of mathematical problems in science and engineering, or internally in math.**
  - **Maxwell's analysis of the centrifugal governor**
  - **Finite element methods in PDE**
  - **Noether's theorem (relationship between laws and invariances)**
  - **Category theory in algebraic topology**

Theodore von Kármán, Josiah Willard Gibbs Lecture. 1939

Due to this failure of the method we do not get an answer for one of the fundamental questions of the hydrodynamics of real fluids, that is : What is the flow pattern of a real fluid around a submerged body in the limiting case $\nu \longrightarrow 0$? As a matter of fact this problem is still not solved. Consider, for example, two-dimensional flow around a circular cylinder. We are not able to decide whether the flow pattern for $\nu \longrightarrow 0$ *approaches* ***the potential flow of a nonviscous fluid*** *or a stationary* flow pattern consisting of **a vortex-free region and a wake with continuously distributed vorticity**, as suggested by Oseen, or **a nonstationary flow pattern with concentrated vortex columns of alternating circulation**, a flow pattern treated by the present author. It seems that we have here an example in which **the analytical methods are not sufficient, at least at the present time, to solve a problem of purely analytical character.**

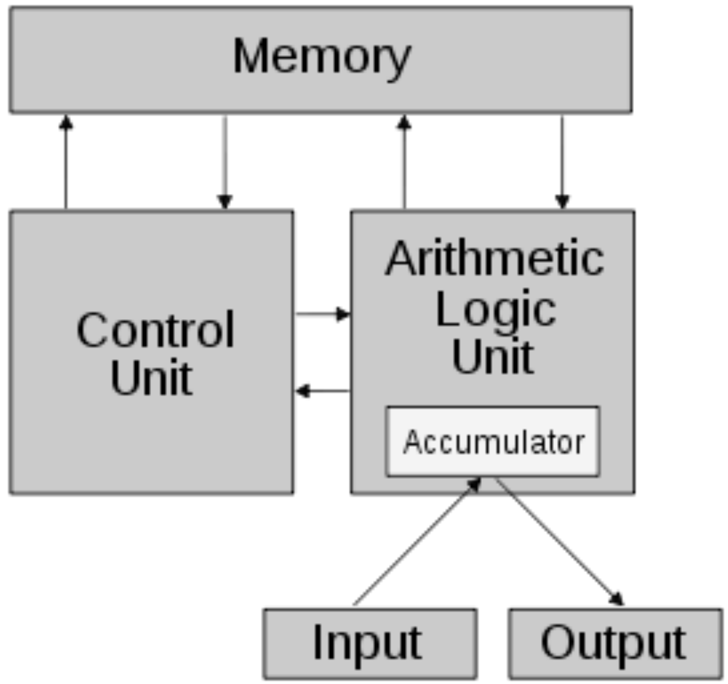
(Theodore von Kármán: *The engineer grapples with nonlinear problems*, Bull. Am. Math. Soc., Volume 46, Number 8 (1940), p.664)

John von Neumann

It was through him [Robert Kent, a senior BRL official] that I was introduced to applied science. Before this I was, apart from some lesser infidelities, essentially a pure mathematician, or at least a very pure theoretician. Whatever else may have happened in the meantime, **I have certainly succeeded in losing my purity**.

William Aspray. John von Neumann and the Origins of Modern Computing. The MIT Press, Cambridge, Massachusetts, 1990, p. 26

Von Neumann Architecture



The term von Neumann architecture arose from von Neumann's paper ***First Draft of a Report on the*** *EDVAC* dated 30 June 1945, where von Neumann gives a detailed description of the logical and technical structure of the EDVAC computer.

The CFvN Weather Forecast

Numerical Integration of the Barotropic Vorticity Equation

By J. G. CHARNEY, R. FJÖRTOFT[1], J. von NEUMANN
The Institute for Advanced Study, Princeton, New Jersey[2]

(Manuscript received 1 November 1950)

Abstract

A method is given for the numerical solution of the barotropic vorticity equation over a limited area of the earth's surface. The lack of a natural boundary calls for an investigation of the appropriate boundary conditions. These are determined by a heuristic argument and are shown to be sufficient in a special case. Approximate conditions necessary to insure the mathematical stability of the difference equation are derived. The results of a series of four 24-hour forecasts computed from actual data at the 500 mb level are presented, together with an interpretation and analysis. An attempt is made to determine the causes of the forecast errors. These are ascribed partly to the use of too large a space increment and partly to the effects of baroclinicity. The rôle of the latter is investigated in some detail by means of a simple baroclinic model.

$$\frac{\partial \eta}{\partial t} + \bar{v} \cdot \nabla \eta = 0$$

$\eta = \varsigma + f =$ absolute vorticity

$\varsigma =$ vertical component of the curl of $\bar{\mathrm{v}}$

$f = 2\Omega \sin\varphi =$ Coriolis parameter

The CFvN Grid

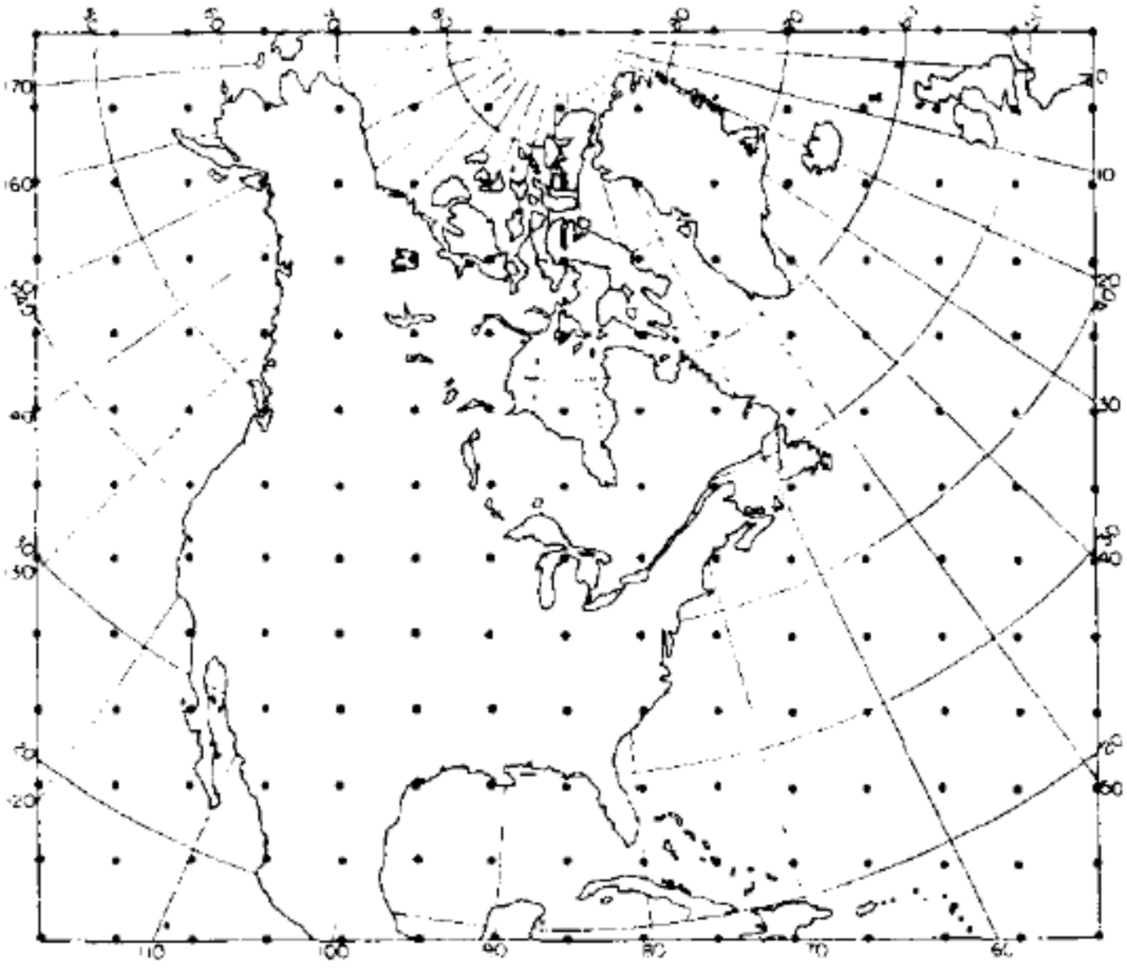


Fig. 1. A typical finite-difference grid used in the computations. A strip two grid intervals in width at the top and side borders and one grid interval in width at the lower border is not shown.

(J. G. Charney, R. Fjörtoft, J. von Neumann: Numerical Integration of the Barotropic Vorticity Equation, Tellus, Vol. 2, No. 4, Nov. 1950, p. 245)

The ENIAC Run

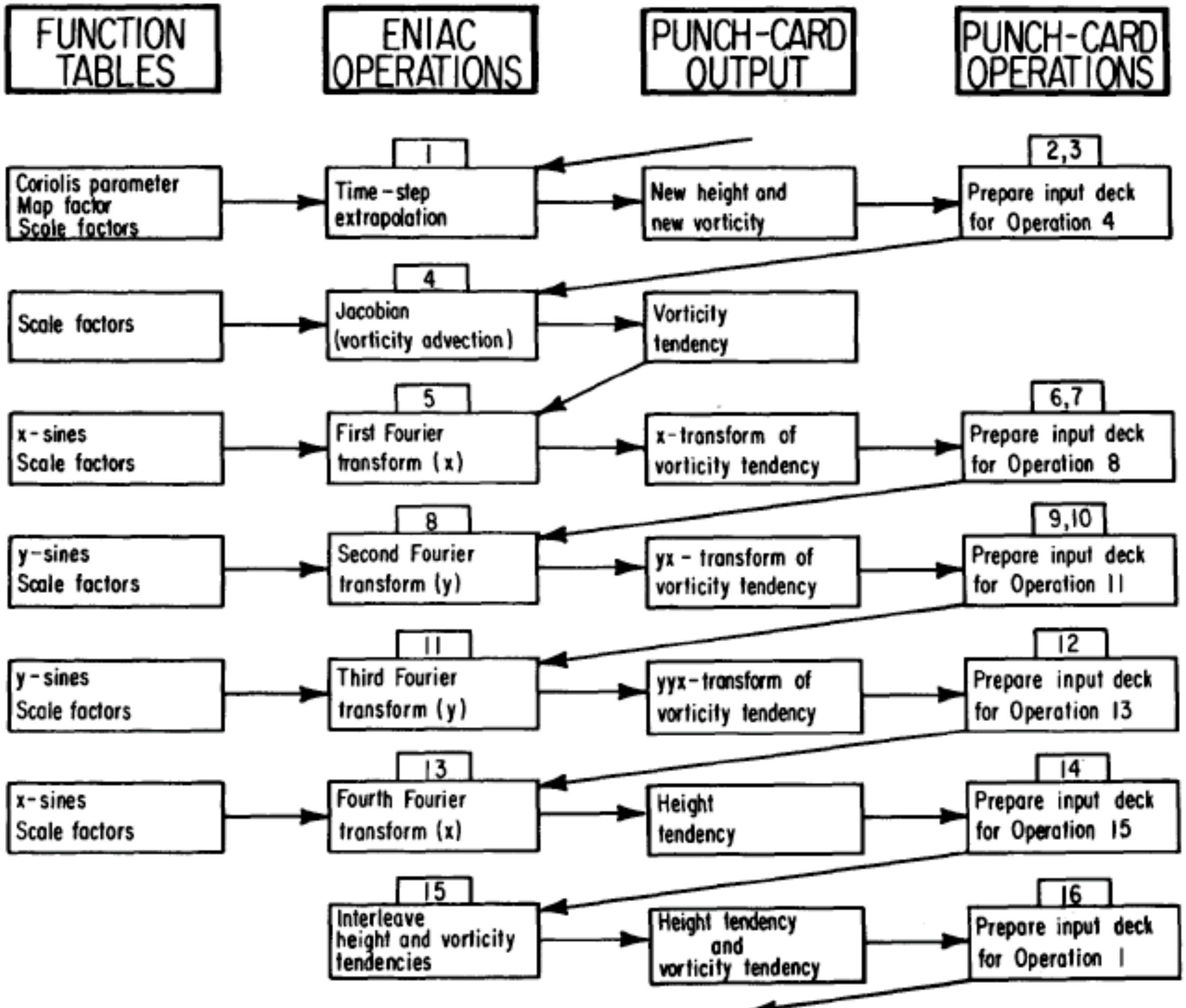
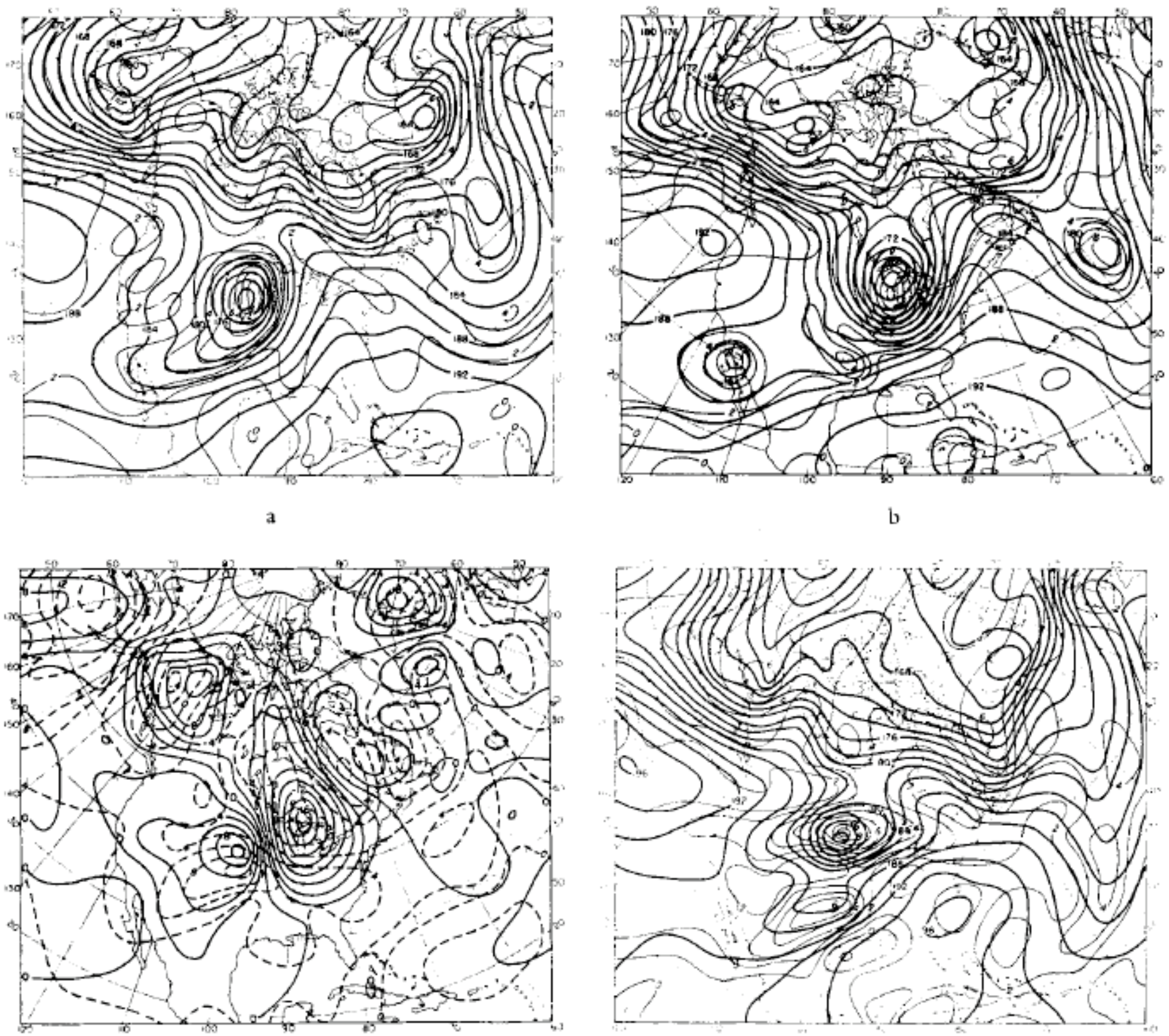


FIG. 6. The 16 operations in each time step of the first numerical weather forecast.



a

b

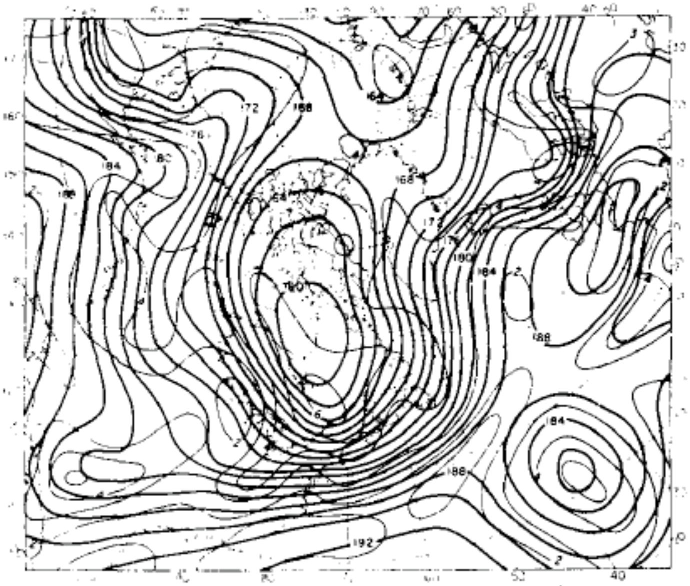
c

d

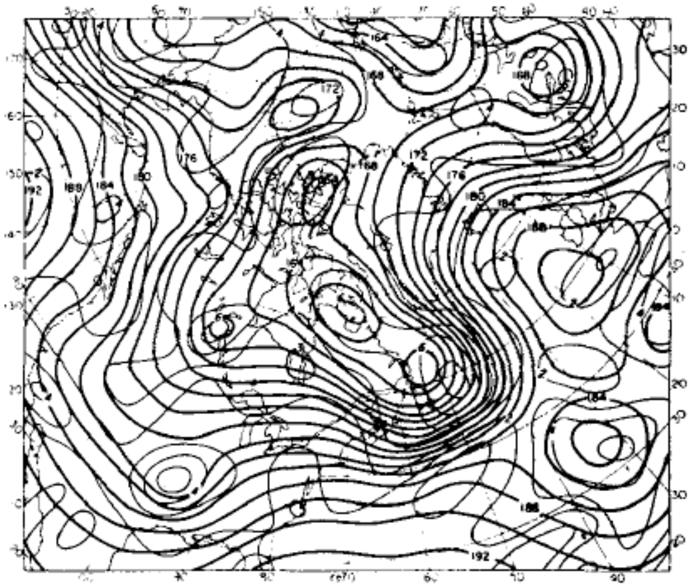
Fig. 2. Forecast of January 5, 1949, 0300 GMT: (a) observed z and η at $t = 0$; (b) observed z and η at $t = 24$ hours; (c) observed (continuous lines) and computed (broken lines) 24-hour height change; (d) computed z and η at $t = 24$ hours. The height unit is 100 ft and the unit of vorticity is $1/3 \times 10^{-4}$ sec^{-1}.

CFvN forecast 1

(J. G. Charney, R. Fjörtoft, J. von Neumann: Numerical Integration of the Barotropic Vorticity Equation, Tellus, Vol. 2, No. 4, Nov. 1950, p. 246)

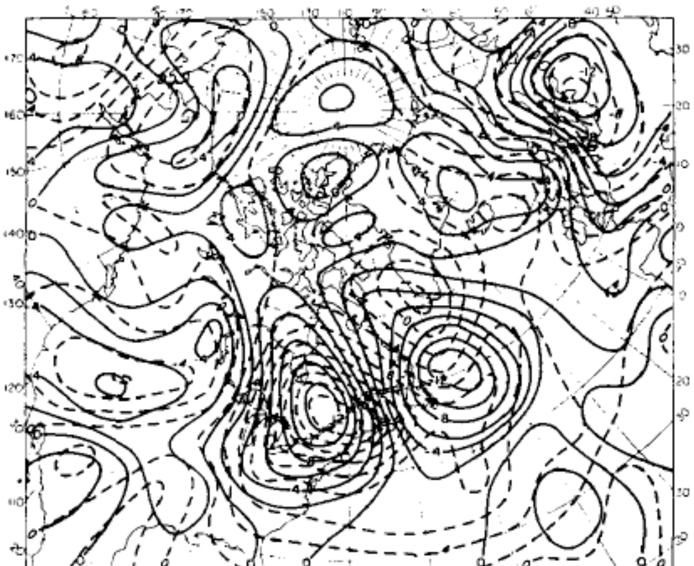


a

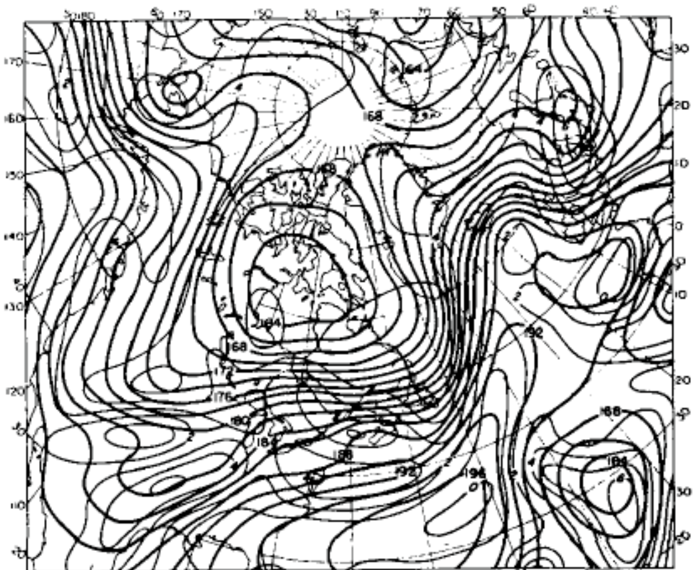


b

CFvN forecast 2

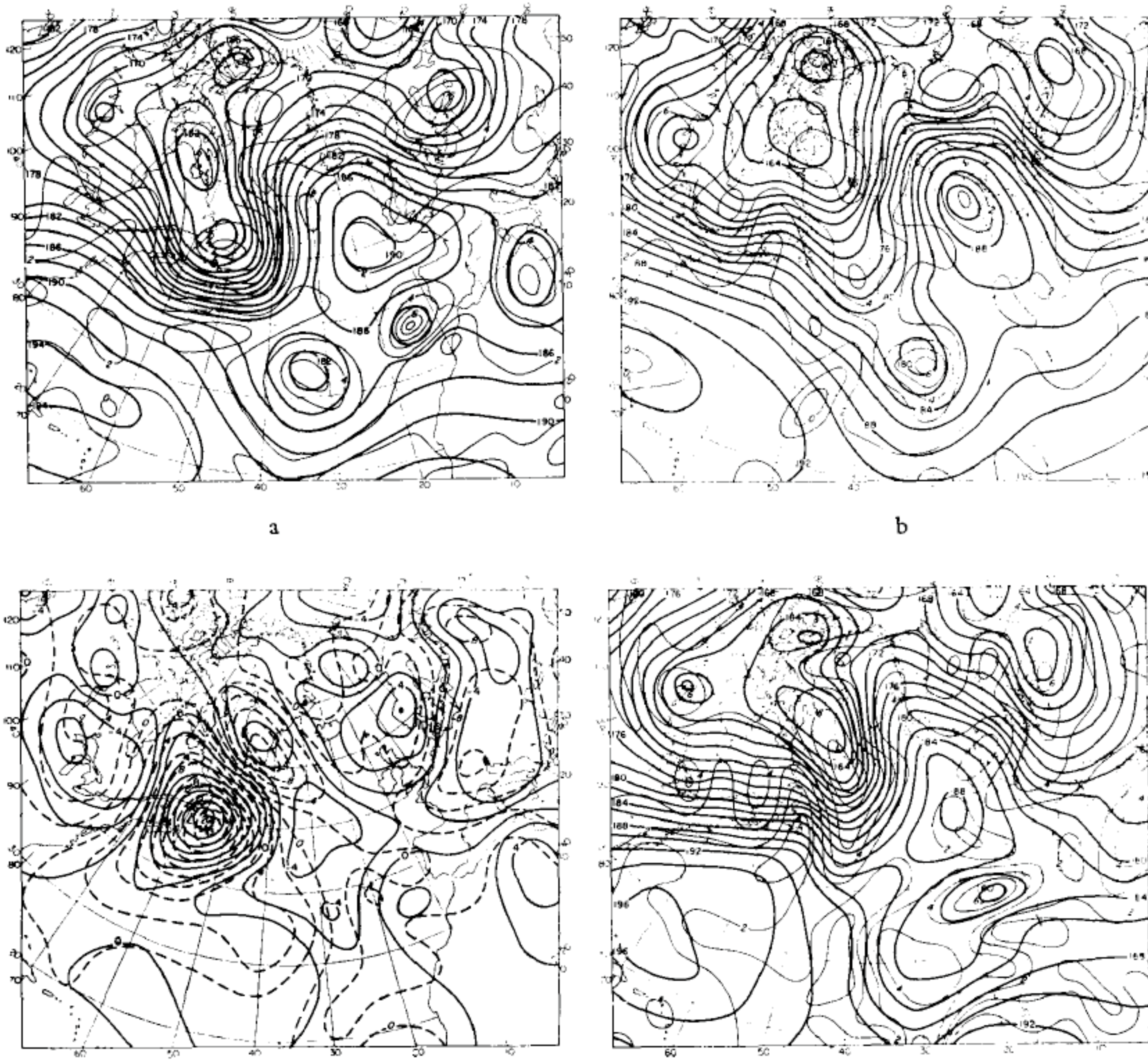


c



d

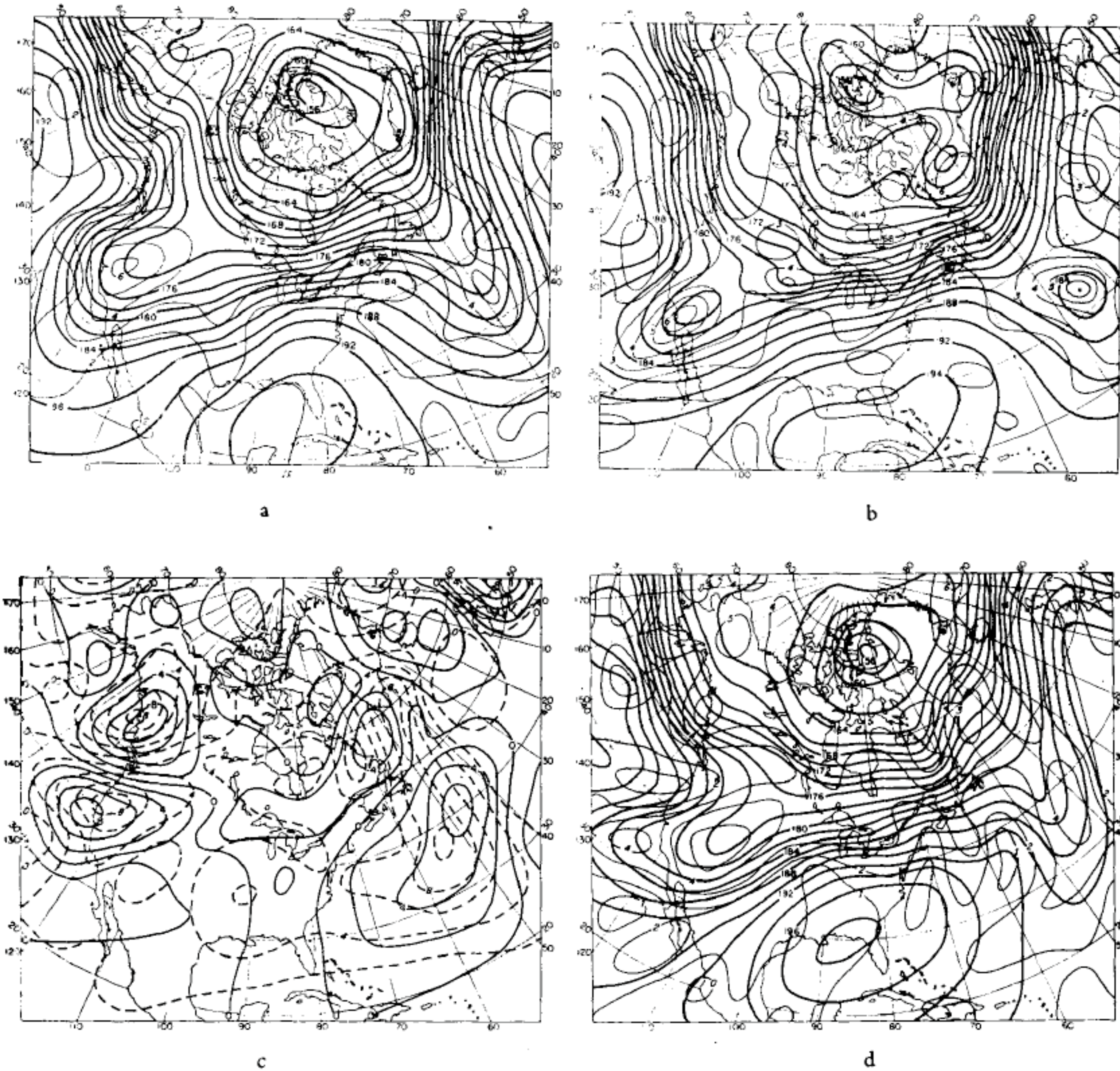
(J. G. Charney, R. Fjörtoft, J. von Neumann: Numerical Integration of the Barotropic Vorticity Equation, Tellus, Vol. 2, No. 4, Nov. 1950, p. 247)



CFvN forecast 3

(J. G. Charney, R. Fjörtoft, J. von Neumann: Numerical Integration of the Barotropic Vorticity Equation, Tellus, Vol. 2, No. 4, Nov. 1950, p. 248)

CFvN forecast 4



(J. G. Charney, R. Fjörtoft, J. von Neumann: Numerical Integration of the Barotropic Vorticity Equation, Tellus, Vol. 2, No. 4, Nov. 1950, p. 249)

Evaluation of Prediction 1 and 2

The forecast of January 5, in which the principal system was an intense cyclone over the United States, was uniformly poor. The forecast gave much too small a displacement of the cyclone and also distorted its shape, and the predictions of the other motions were equally inaccurate. On the other hand, **the January 30 forecast contained a number of good features**. The displacement and amplification of the trough over the United States at about 110^0 W was well predicted, as was the large scale shifting of the wind from NW to WSW and the increase in pressure over eastern Canada. **The displacement of the axis of the major trough over the eastern United States and Canada was correctly predicted, but the strong circulation that developed at its southern extremity was not.** Proceeding eastwards we find that the amplification of the trough over the North Sea together with the characteristic breakthrough of the northwesterly winds and the corresponding destruction over France of the eastern nose of the anticyclone was predicted approximately. This is shown by the agreement of the predicted with the observed height changes over western Europe.

(J. G. Charney, R. Fjörtoft, J. von Neumann: Numerical Integration of the Barotropic Vorticity Equation, Tellus, Vol. 2, No. 4, Nov. 1950, p. 245-246)

Improved Computer Power

During his Starr Lecture, **George Platzman** arranged with IBM to repeat one of the ENIAC forecasts. The algorithm of CFvN was coded on an **IBM 5110**, a desktop machine then called a portable computer or "PC" (having a tiny fraction of the power of a modern PC). The program execution was completed within the hour or so of Platzman's lecture. This implies a 24-fold speedup over the best rate achievable for ENIAC. The program eniac.m was run on a Sony Vaio (model VGN-TX2XP) with MATLAB version 6. **The main loop of the 24-h forecast ran in about 30 ms**. One may question the precise significance of the time ratio—about three million to one—but it certainly indicates the dramatic increase in computing power over the past half-century.

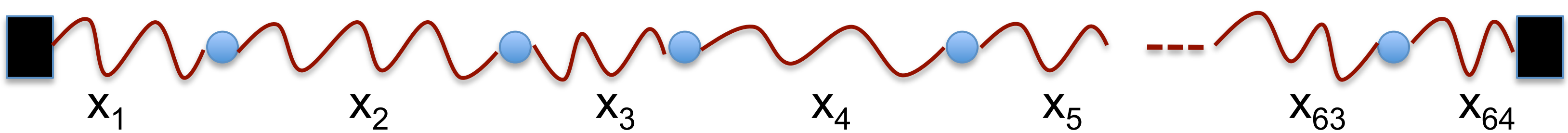
(Peter Lynch: The ENIAC Forecast, a Re-creation, Bull. Am. Meteorological Society, January 2008)

S.M. Ulam about Fermi

After the war, during one of his frequent summer visits to Los Alamos**, Fermi became interested in the development and potentialities of the electronic computing machines**. He held many discussions with me on the kind of future problems which could be studied through the use· of such machines. We decided to **try a selection of problems for heuristic work where in absence of closed analytic solutions experimental work on a computing machine would perhaps contribute to the understanding of properties of solutions.** This could be particularly fruitful for problems involving the **asymptotic-long time or "in the large" behavior of non-linear physical systems**. In addition, such experiments on computing machines would have at least the virtue of having the postulates clearly stated. This is not always the case in an actual physical object or model where all the assumptions are not perhaps explicitly recognized.

Fermi expressed often a belief that **future fundamental theories in physics may involve non-linear operators and equations**, and that it would be useful to attempt practice in the mathematics needed for the understanding of non-linear systems.

The experiment



$$\ddot{x}_i = (x_{i+1} + x_{i-1} + 2x_i) + \alpha\left[(x_{i+1} - x_i)^2 + (x_i - x_{i-1})^2\right]$$

$$or$$

$$\ddot{x}_i = (x_{i+1} + x_{i-1} + 2x_i) + \beta\left[(x_{i+1} - x_i)^3 + (x_i - x_{i-1})^3\right]$$

The results of the calculations (performed on the old MANIAC machine) were interesting and quite surprising to Fermi. He expressed to me the opinion that they really constituted a little discovery in providing intimations that **the prevalent beliefs in the universality of "mixing and thermalization" in non-linear systems may not be always justified.**

Modal Representation

$$H(q,p)=\sum_{j=1}^{n+1}\left[\tfrac{1}{2}(p_j^2+(q_j-q_{j-1})^2+\tfrac{\alpha}{3}(q_j-q_{j-1})^3\right]$$

$$a_k=\sqrt{\frac{2}{n+1}}\sum_{j=1}^{n}q_j\sin\left(\frac{jk\pi}{n+1}\right)$$

$$q_j=\sqrt{\frac{2}{n+1}}\sum_{j=1}^{n}a_k\sin\left(\frac{jk\pi}{n+1}\right)$$

$$H(a,\dot{a})=\frac{1}{2}\sum_{k=1}^{n}(\dot{a}_k^2+\omega_k^2a_k^2)+\alpha\sum_{j,k,l=1}^{n}A_{j,k,l}q_jq_kq_l \qquad \omega_k=2\sin\left(\frac{k\pi}{2(n+1)}\right)$$

Fermi, Pasta and Ulam's observations

Instead of a gradual, continuous flow of energy from the first mode to the higher modes, all of the problems show an entirely different behavior. Starting in one problem with a quadratic force and a pure sine wave as the initial position of the string, we indeed observe initially a **gradual increase of energy in the higher modes as predicted** (e.g., by Rayleigh in an infinitesimal analysis). Mode 2 starts increasing first, followed by mode 3, and so on**. Later on, however, this gradual sharing of energy among successive modes ceases. Instead, it is one or the other mode that predominates.** For example, mode 2 decides, as it were, to increase rather rapidly at the cost of all other modes and becomes predominant. At one time, it has more energy than all the others put together! Then mode 3 undertakes this role. **It is only the first few modes which exchange energy among themselves and they do this in a rather regular fashion**. Finally, at a later time **mode 1 comes back to within one percent of its initial value so that the system seems to be almost periodic**. All our problems have at least this one feature in common. **Instead of gradual increase of all the higher modes, the energy is exchanged, essentially, among only a certain few. It is, therefore, very hard to observe the rate of "thermalization" or mixing in our problem, and this was the initial purpose of the calculation.**

(E. Fermi, J. Pasta, and S. Ulam: Studies of non linear problems, Document U-I940 (may 1955), p. 981)

Results

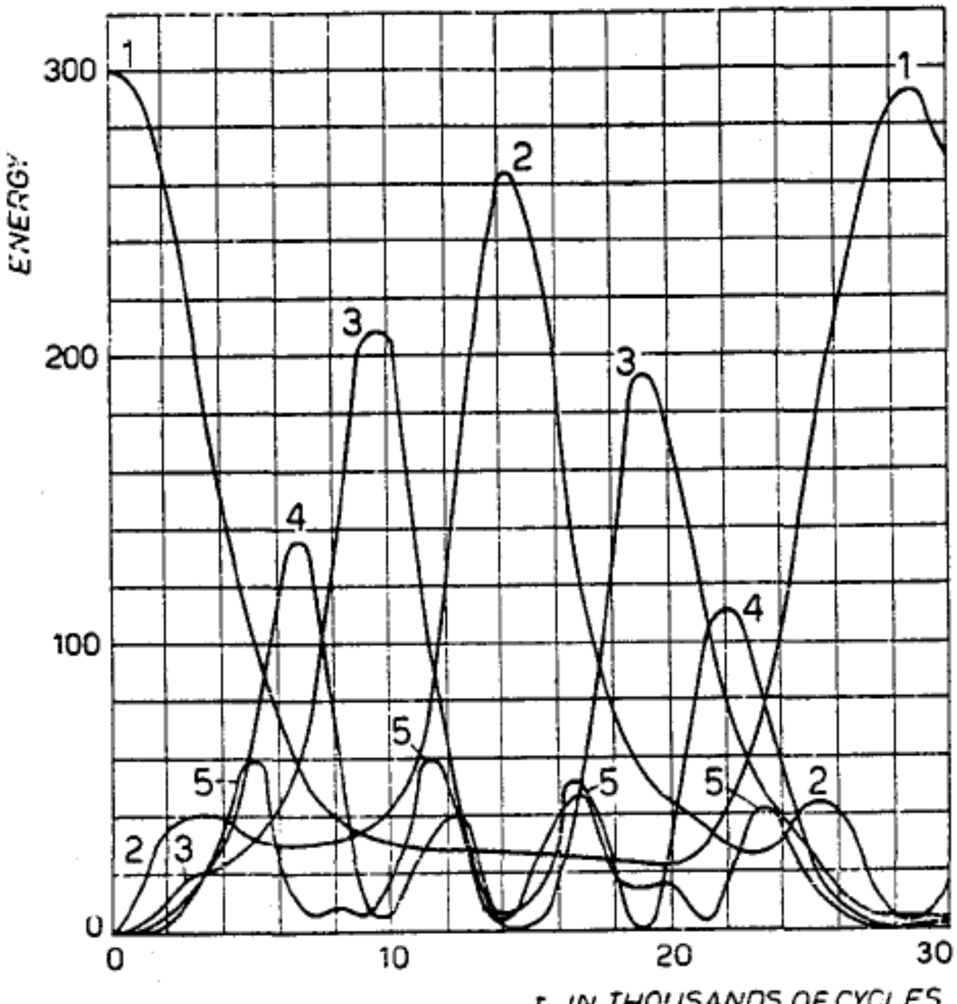


Fig. 1. – The quantity plotted is the energy (kinetic plus potential in each of the first five modes). The units for energy are arbitrary. $N = 32$; $\alpha = 1/4$; $\delta t^2 = 1/8$. The initial form of the string was a single sine wave. The higher modes never exceeded in energy 20 of our units. About 30,000 computation cycles were calculated.

Later calculations

In 1961, on more modem and faster machines, the original problem was considered for still longer periods of time. It was found by J. Tuck and M. Menzel that after one continues **the calculations from the first “ return" of the system to its original condition the return is not complete**. The total energy is concentrated again essentially in the first Fourier mode, but the remaining one or two percent of the total energy is in higher modes. If one continues the calculation, at the end of the next great cycle the error (deviation from the original initial condition) is greater and amounts to perhaps three percent. Continuing again one finds the deviation increasing - **after eight great cycles the deviation amounts to some eight percent; but from that time on an opposite development takes place!** After eight more i.e., sixteen great cycles altogether, the system gets very close - better than within one percent to the original state! This **super-cycle** constitutes another surprising property of our non-linear system.

The challenge of Numerical Analysis

The ENIAC and similar early computers have very small memories for intermediate results but was able to do multiplications - which was the measure of computational complexity at that time - extremely fast. So**, as numerical analysts up to this time considered multiplications to be slow and expensive and storage cheep, they now lived in a world where multiplications were extreme fast but storage very pour**. A consequence of this was that it became necessary to reexamine existing algorithms, and suddenly it became possible to solve partial differential equations numerically.

Numerical Analysis

Numerical analysis is the area of mathematics and computer science that creates, analyzes, and implements algorithms for **solving numerically the problems of continuous mathematics**. Such problems originate generally from real-world applications of algebra, geometry, and calculus, and they involve variables which vary continuously. These problems occur throughout the natural sciences, social sciences, medicine, engineering, and business.

K. E. Atkinson. Numerical analysis. Scholarpedia, 2(8):1–2, 2007.

Numerical Analysis Since the 1940's

Beginning in the 1940's, the growth in power and availability of digital computers has led to an increasing use of realistic mathematical models in science, medicine, engineering, and business; and numerical analysis of increasing sophistication has been needed to solve these more accurate and complex mathematical models of the world. **The formal academic area of numerical analysis varies from highly theoretical mathematical studies to computer science issues involving the effects of computer hardware and software on the implementation of specific algorithms.**

K. E. Atkinson. Numerical analysis. Scholarpedia, 2(8):1–2, 2007.

Reasoning Styles

By a scientific reasoning style we understand a comprehensive transformation of the way in which science is being understood and performed. The transformation covers the following elements:

1. The transformation concerns several scientific disciplines.

2. New institutions are formed that epitomize the new directions.

3. It leads to new social organizations of the scientific practice.

4. It leads to fundamental ontological and epistemological changes: New types of objects, evidence, classifications, laws or modalities, and ways of expressing scientific facts.

The Probabilistic Reasoning Style

The numerous statistical societies founded in the 1830s are some of the new institutions associated with what Hacking called the probabilistic revolution. The avalanche of numbers gave a different feel to the world: it had become quantified and numbers and statistics ruled it. It was Mr. Gradkin's world. Concomitantly, the previously dominant determinist Weltanschauung became replaced by a view of the world in which probability and chance played an ever increasing role. The result was **the emergence of a new statistical style, constituted by a plethora of abstract statistical entities and governed by autonomous statistical laws, which are 'used not only to predict phenomena but also to explain [them]'.**

(Sam Schweber and Matthias Wächter. Complex systems, modelling and simulation. Stud. Hist. Phil. Mod. Phys., 31, 2000, p. 584)

Computational Science as a New Reasoning Style

We are witnessing another Hacking-type revolution, in which the computer is the central element - in the same sense that the steam engine was the central element in the first industrial revolution and that factories driven by steam power and steam locomotives and railroads transformed the economic and social landscape. That the computer has similarly generated a sweeping transformation of the social, material, economic and cultural context is evident - think only of the transformation of the workplace and the novel routinisations that the computer has introduced, of e-commerce, of the new classes of professionals, etc.

(Sam Schweber and Matthias Wächter. Complex systems, modelling and simulation. Stud. Hist. Phil. Mod. Phys., 31, 2000, p. 585)

Computational Science

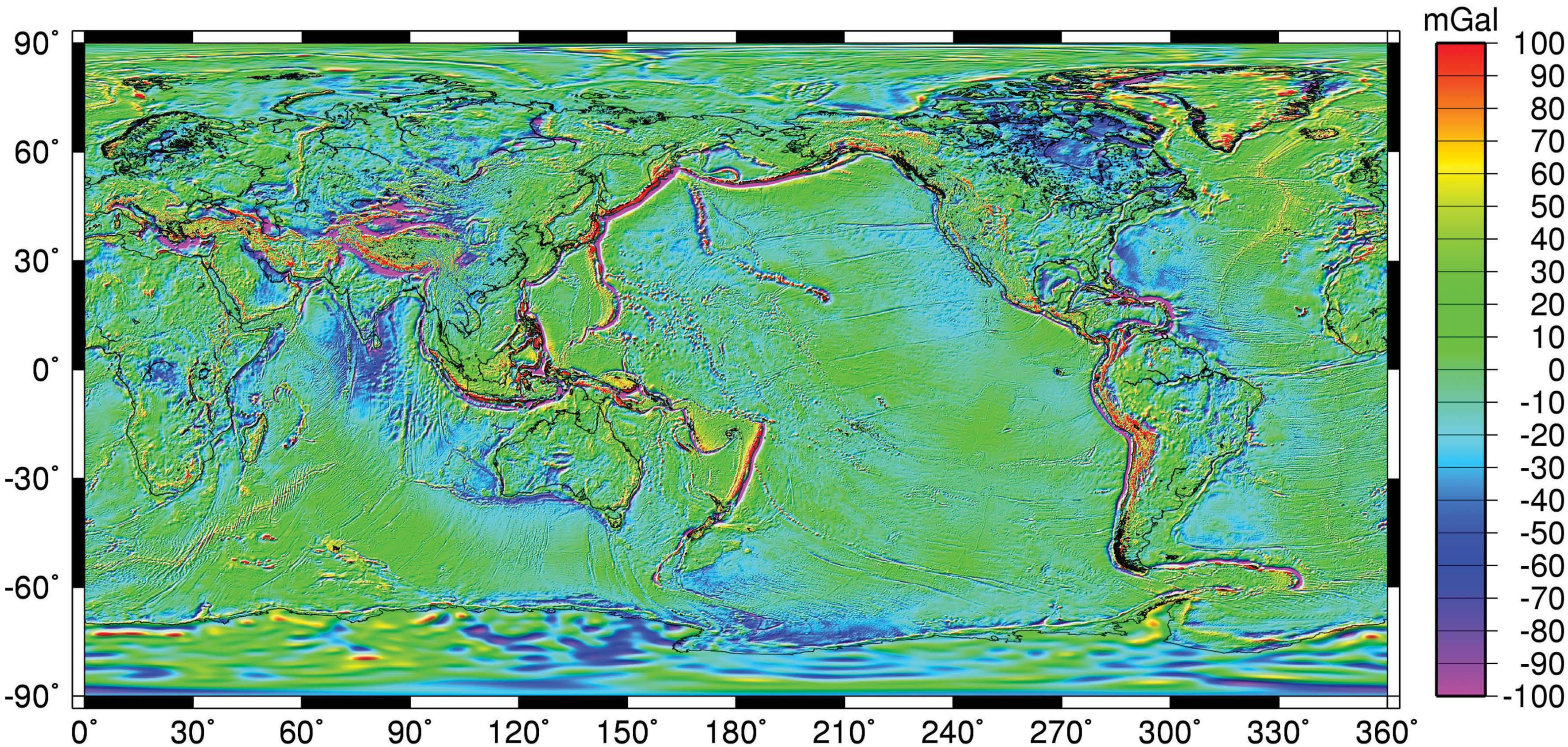
We are in the midst of a computational revolution that will change science and society as dramatically as the agricultural and industrial revolutions did. The discipline of computational science is significantly affecting the way we do hard and soft science.

…

Supercomputers with ultrafast, interactive visualization peripherals have come of age and provide a mode of working that is coequal with laboratory experiments and observations and with theory and analysis. We can now grapple with nonlinear and complexly intercoupled phenomena in a relatively short time and provide insight for quantitative understanding and better prediction. In the hands of enthusiastic and mature investigators, intractable problems will recede on a quickened time scale in this computationally synergized environment.

(Norman J. Zabusky: Physics Today, October 1987, p. 25)

Free-air Gravity Anomalies From the Earth Gravitational Model 2008 (EGM2008)



Free-air gravity anomalies computed from EGM08, averaged over 5 arc-minute by 5 arc-minute cells on the surface of the Earth. A gravity anomaly is the difference of actual (observed) gravity from a nominal (theoretical) value. The unit is “milliGal” (denoted mGal, where 1 mGal = 10^{-5} ms^{-2}), which corresponds approximately to 1 part per million of the gravity acceleration sensed by an observer on the Earth’s surface. Notice the numerous geophysical features that are revealed, such as oceanic trenches, ridges, subduction and fracture zones, and seamount chains.

Conflict between Computer Science and Numerical Analysis

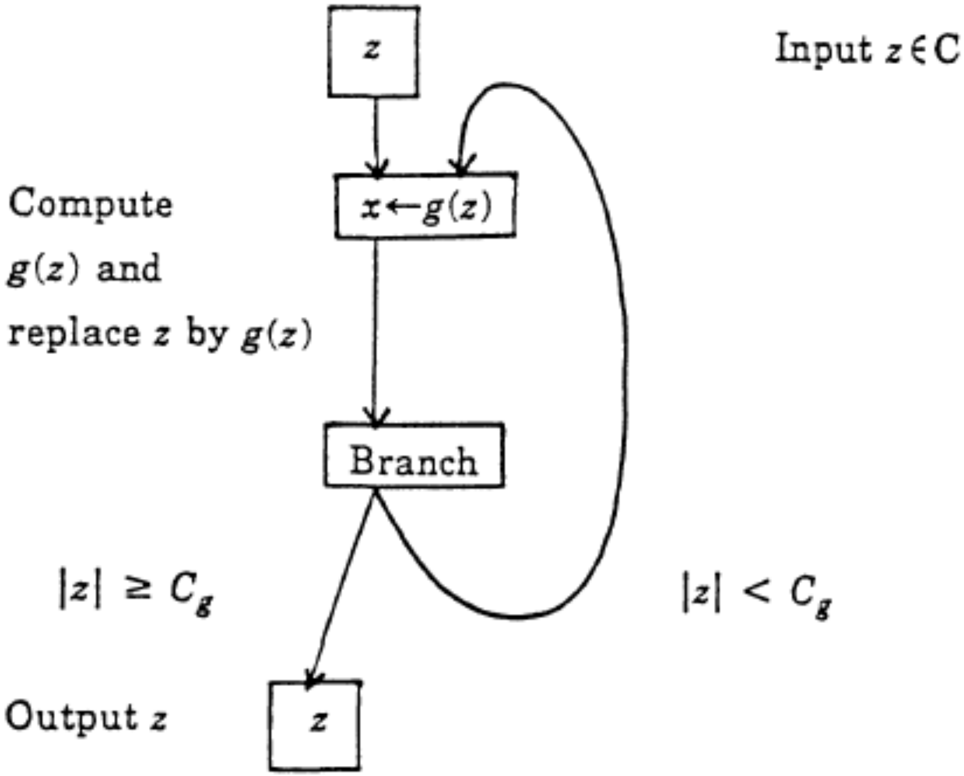
There is a substantial conflict between theoretical computer science and numerical analysis. These two subjects with common goals have grown apart. For example, computer scientists are uneasy with calculus, whereas numerical analysis thrives on it. On the other hand numerical analysis see no use for the Turing machine.

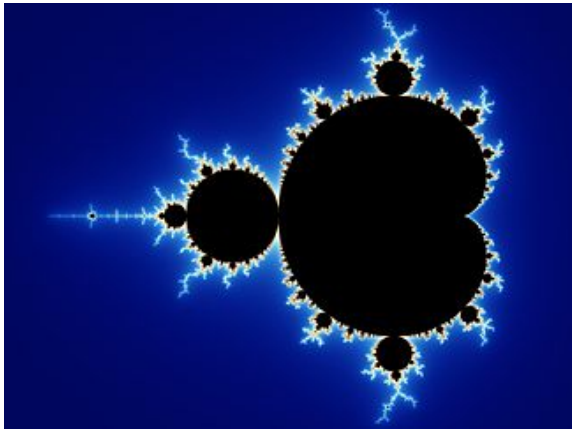
…

A major obstacle to reconciling scientific computation and computer science is the present view of the machine, that is, the digital computer. As long as the computer is seen simply as a finite or discrete object, it will be difficult to systematize numerical analysis. We believe that the Turing machine as a foundation for real number algorithms can only obscure concepts.

(Lenore Blum, Felipe Cucker, Michael Shub, and Steve Smale. Complexity and real computation. Springer-Verlag, New York, 1998, p. 23)

Machine corresponding to the polynomial map $g: C \rightarrow C$





Ω_M: The halting set of M = all z in C where $g^k(z) \rightarrow \infty$ when $k \rightarrow \infty$

If $g(z)=z^2+c$, then Ω_M is R.E but not decidable over R.

Thank you for your attention