Philosophy of mathematics in different fields

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Cultures of Mathematics and Logic

Structure of the talk

1. The philosophy of mathematicians
   - Philosophy in mathematical journals
   - Philosophical views of mathematicians
2. Foundational concerns in mathematical textbooks
   - What was studied?
   - Results
   - What should be done?
3. Conclusion

Structure of the talk

Philosophy in mathematical journals

- Starting point: is there a way to determine the philosophical views of mathematicians by studying their mathematical work?
- First stage: a wide selection of articles in top-ranked general mathematical journals were studied with the focus on four main questions:
  - Are there explicit philosophical views in them?
  - Is there a valid way to see implicit philosophical views?
  - What are the background theories used?
  - Are there differences in methodology and rigour?
- Branches of mathematics with obvious connections to philosophy, i.e. mathematical logic and set theory, were not studied.

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Philosophy in mathematical journals

- The results were almost uniformly uninteresting for all four questions.
- Explicit philosophical statements are nearly non-existing in mathematical journals.
- There is a possibility to see implicit philosophical content in the language of mathematics, but this is highly questionable.
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The Language of mathematics

- The language in mathematical journals would suggest immediate philosophical relevance.
- Phrases like "there exists a number/metric/field/etc." are often used.
- But this can mean equally well belief in a Platonic existence of the object or the mere validity of a construction based on axioms.
- To read philosophical importance in such language is unacceptable.
- Most mathematicians seem to be *working realists* (Shapiro 1997): they work as if mathematics had an objective mind-independent subject matter.
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Hersh on working realism

Hersh: (1997, 39) "[a mathematician is] a Platonist on weekdays, a formalist on weekends. On weekdays, when doing mathematics, he's a Platonist, convinced he's dealing with an objective reality whose properties he's trying to determine. On weekends, if challenged to give a philosophical account of the reality, it's easiest to pretend he doesn't believe it. He plays formalist, and pretends mathematics is a meaningless game."

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Background theories

- What is the philosophical status of background theories (logic, set theory)?
- Terence Tao:
  "I recently came across the phenomenon of *nonfirstorderisability* in mathematical logic: there are perfectly meaningful and useful statements in mathematics which cannot be phrased within the confines of first-order logic (combined with the language of set theory, or any other standard mathematical theory). To phrase such statements rigorously, one must use a more powerful language such as second-order logic instead. This phenomenon is very well known among logicians, but I had not learned about it until very recently, and had naively assumed that first-order logic sufficed for "everyday" usage of mathematics." (2008, 79)

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- In analysis, it is common to see expressions like: "...assuming N is chosen sufficiently large depending on ϵ and δ chosen sufficiently small depending on N..."
- In a direct first-order formulation of the sentence, the variable δ would depend on both N and ϵ.
- Thus the mathematician uses English to bring in a second-order sentence capable of expressing the desired dependencies.
- In this way, the background theory is often not explained, but rather chosen *ad hoc* to fit the purpose.
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- What logicians and philosophers consider the standard of rigour for mathematics is often very different from that of working mathematicians.
- This is not to say that mathematical proofs are not rigorous. Rather, they are that by their own, more informal, standards.
- For the most part, there is nothing wrong with that. Needlessly close connection to foundational matters could make mathematical research less fruitful. At least as philosophers we should be very careful about claiming otherwise.
- Nevertheless, *knowledge* of such foundational matters could not hurt any mathematician.
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1. The philosophy of mathematicians
   - Philosophy in mathematical journals
   - Philosophical views of mathematicians
2. Foundational concerns in mathematical textbooks
   - What was studied?
   - Results
   - What should be done?
3. Conclusion

Empirical studies

- So how can we study the philosophical views of mathematicians?
- There have been some empirical studies on the subject:
  - Grigutsch & Törner (1998): 85 percent of the subject group of university math teachers in German-speaking countries are only moderately or weakly Platonist.
  - But what was *strong* Platonism in that study? Here are two sample questions:

  "God is a child, and he did mathematics as he began to play. It is the godliest of games among mankind."
  "When the laws of mathematics are related to reality they are not secure, and when they are secure, they are not related to reality."
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- In the study of Wilhelmus (Löwe, Müller & Wilhelmus (2007)), 82,4% of mathematicians answered that mathematical knowledge is objective (the conclusion of the paper is the opposite - see Pantsar (2012) for analysis of it).
- How can we account for such differences?
- In such studies, we must make sure that we are absolutely clear about what is meant by the philosophical concepts like Platonism, existence, objectivity, knowledge, etc.
- The studies so far seem to include a strong possibility of confusion about them.
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- Two ways:
  - As mathematicians: providing philosophers with their subject matter.
  - As philosophers: by philosophical argumentation.
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- If we think that philosophical problems about mathematics are *bona fide* problems, we must believe that they are solved by argumentation rather than intuition.
- In this, the contribution of mathematicians would be of crucial importance. They have both a deep knowledge of the subject matter as well as a familiarity with the methodology.

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Structure of the talk

Foundational concerns

- What is meant by foundational concerns?
  - explanations of logic
  - explanations of set theory
  - explanations of proof methods or axiomatic systems
  - philosophical discussions of mathematical concepts
  - some historical explanations

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- Undergraduate and graduate level textbooks in English.
- A sample of 4-6 textbooks currently in use for each branch of mathematics was selected.
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Structure of the talk

1. The philosophy of mathematicians
   - Philosophy in mathematical journals
   - Philosophical views of mathematicians
2. Foundational concerns in mathematical textbooks
   - What was studied?
   - Results
   - What should be done?
3. Conclusion

What kind of foundational concerns were included?

- A sample case from Hatcher's *Algebraic Topology*, after proving the Fundamental Theorem of Algebra and Brouwer's fixed-point theorem (p. 32):

"These proofs are all arguments by contradiction, and so they show just the existence of fixed points without giving any clue as to how to find one in explicit cases. Our proof of the Fundamental Theorem of Algebra was similar in this regard. There exist other proofs of the Brouwer fixed point theorem that are somewhat more constructive, for example the elegant and quite elementary proof by Sperner in 1928, which is explained very nicely in [Aigner-Ziegler 1999]."

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- The student is lead to the meta-mathematical issue of there being essentially two different kinds of proofs.
- But she is not given any general explanation about non-constructive proofs (most likely partly because Hatcher doesn't use footnotes). There is a link to foundational concerns as well as the philosophy of mathematics, but it is kept quite minimal.
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- During the research, one thing became obvious early on. During the recent decades, there has been a visible to change to a more explanatory mode of exposition in mathematical textbooks.
- Examples: the classic Hardy & Wright et al (1st edition 1938): *An Introduction to the Theory of Numbers* is much more formal than Jones & Jones (1998): *Elementary Number Theory*.
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- Other findings:
  - Probability textbooks are a mixed bunch. Curiously, those that were more applied in nature included much more foundational and philosophical concerns (e.g. Jaynes & Bretthorst: *Probability Theory*).
  - Textbooks on number theory do not mention the method of defining natural numbers as sets. Overall, they are among the most formal and least foundational textbooks.
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Structure of the talk

Do mathematicians need foundations?

- I do not want to suggest mathematics should be radically changed; that we should take a foundationalist program like logicism and remodel higher mathematical education based on that.
- What I do want to suggest is that math students would benefit from having a wider and deeper understanding of their subject. As a desirable side effect, this would also facilitate the interaction between mathematicians and philosophers.
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- Do this without "dumbing down" the foundations. Teach naive set theory, but mention Russell's paradox in a footnote. Teach the axiomatic method but mention Gödel. Perhaps even teach the difference between first- and second-order theories and mention their differences in arithmetical models.
- It is my hypothesis, backed with a lot of anecdotal evidence, that math students are more interested in foundational matters when presented in this way. Set theory may not be exciting, but Russell's paradox is. Logic may seem obvious, but the incompleteness of second-order logic is surprising.

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How to do all this?

- I do not pretend to have ready-made answers about how make a, say, topology textbook as foundationally interesting as possible. But from the existing textbooks, a lot of good ideas can be seen.
- Pinter: *Abstract Algebra* is very discussional in its form with a lot of emphasis in explaining what abstract algebra is philosophically. The abstractness of negative numbers, complex numbers etc. are discussed, as are the axiomatic method and how integers can be treated algebraically, as an ordered set, or by recursion theory.

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"[A]n important aspect of axiomatic mathematics is this: when we capture mathematical facts in an axiomatic system, we never try to reproduce the facts in full, but only that side of them which is important or relevant in a particular context. This process of *selecting what is relevant* and disregarding everything else is the very essence of abstraction."

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"We adopt. as most mathematicians do, the naive point of view regarding set theory. We shall assume that what is meant by a *set* of objects is intuitively clear, and we shall proceed on that basis without analyzing the concept further. (...) It is unfortunately true that careless use of set theory, relying on intuition alone, can lead to contradictions. Indeed, one of the reasons for the axiomatization of set theory was to formulate rules for dealing with sets that would avoid these contradictions. Although we shall not deal with the axioms explicitly, the rules we follow in dealing with sets derive from them. (...) At some point of your studies, you may wish to

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- The reader is lead to other courses, which is totally fine, but he has got the spark that there is more to set theory than he thought.
- Again the question is: how could such a Chapter 0 make the textbook *worse*?

How to do all this?

- The reader is lead to other courses, which is totally fine, but he has got the spark that there is more to set theory than he thought.
- Again the question is: how could such a Chapter 0 make the textbook *worse*?

Conclusion

- It is hard to determine the philosophical views of mathematicians from their mathematical work. But this is not what we should be after. There is no reason to believe mathematicians have a privileged access to philosophical answers.
- Rather, we should try to educate mathematicians in foundational and philosophical matters, in which case, due to background, they would play an extremely valuable role in philosophical discussion. This could also be beneficial for mathematicians in their own work, as well.

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- How should this be done? Even if they don't take courses on logic or set theory, every math student's university education should include foundational matters. Textbooks in subjects like algebra, topology and analysis should include more meta-mathematics. Introducing exceptional aspects like paradoxes and unintuitive results would make the students motivated to find out more about the subjects.
- In conclusion: mathematicians don't work for philosophers, but all future mathematicians should be given a chance to get acquainted with foundational and philosophical matters. This does not require a radical change: in algebra and topology textbooks this is already done to some degree. This development must be encouraged, in the same way that abandoning strict formalism in textbooks has been encouraged in the recent decades.

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