SOME ASPECTS OF MATHEMATICAL PLURALISM

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One of the ancient intellectual activities pursued from the beginning of civilization is mathematics.

However, at its beginning mathematics was not a purely intellectual act, it was embedded in the requirements of daily life.

"As there is the crest in the head of a peacock or the jewel in the head of a snake so is mathematics on the head of all knowledge."

- ***Vedanga Jyotish***

(c. 1180 B.C.)

Let us have a look at the list:

Babylonia: ..... 2000 B.C.

[O. Neugebauer – The Exact Science in Antiquity]

Egypt: ..... 1800 B.C.

[Thomas Eric Peet – The Rhind Mathematical Papyrus]

China: ..... 2700 B.C.

[Yoshi Mikani – The Development of Mathematics in China and Japan]

India: ..... 3000 B.C. (?)

[Sulva Sutra – 800 B.C.]

In an interview in 2008, noted mathematician Yuri Manin says the following two sentences simultaneously

- "Mathematics is a part of culture, in the broad sense of the term and not part of industry or service or something of that sort."
- "I think that people engaged in research in mathematics today are doing so the same way it was done 200 years ago."

"We do not choose mathematics as our profession, it chooses us: Interview with Yuri Manin" – by Mikhail Gelfand, Sept, 30, 2008.

On the other hand, in R. L. Wilder's opinion:

- "What is not mathematics today (because of its lack of relation to the existing mathematics) can one day become mathematics."
- "A mathematician of the early part of this century would probably be appalled by the mathematics of today."

R. L. Wilder, Introduction to the foundations of mathematics, John Wiley & Sons, 1965.

Our own studies lead towards culture dependency of mathematics and the position (Wilder's) that mathematics in content form and practice is ever changing - perhaps in a slower pace, relative to other fields.

A careful glance at present mathematics would reflect a great deal of change causing thereby a great diversity not only in the external but also in essence (as opposed to Manin's view).

In this presentation we focus on some expressions of this diversification.

Diversifications may be observed at least from the following angles:

1. Differences in procedures
2. Differences in underlying axioms
3. Differences in underlying logics
4. (other) Differences in the foundation
5. Difference in interpretations
6. Differences due to culture, time and history

1. Differences in procedures:

The classicist (Platonic) adopt procedures in mathematical investigations that comply with the viewpoint that mathematical entities are there, outside the human mind and the task of a mathematician is to discover their laws. Mathematical entities are not creations of human minds, the latter serves as the epistemic instrument to derive knowledge of these objects.

Manin declares in the said interview:

> "I am an emotional Platonist (not a rational one: there are no rational arguments in favour of Platonism.)"

On the other hand, non-classicists are of many shades. One variety of them engage in constructing mathematical entities. [Intuitionists/ Constructivists]

Thus a kind of diversification stems from the approach taken regarding the ontic status of mathematical entities and depending upon the nature of involvement of human mind in mathematical activities.

Within constructivists diversification is observed on the ways of construction of mathematical entities.

- In 1974, the American Mathematical Society sponsored a meeting to evaluate and to explore the consequences of Hilbert problems. One of the interesting implications of the Congress was that a new list of mathematical problems was proposed. According to Felix Browder, the editor of the proceedings, this list was initiated by Jean Dieudonnne through correspondence with mathematicians through out the world. The first problem of this new list deals with the foundations of mathematics which was mentioned by Manin as

  "We need a creative approach not just a critical one."

and

  "We should consider possibilities of developing a totally new language to speak about infinity."

1. Differences in procedures:

Among the non-classicists, diversification arise from many other angles as will be mentioned in some of the following slides.

Difference in procedures also generate from the perspective of developing mathematics either solely as a tool or as autonomous enterprise of abstract forms.

2. Differences in the underlying axioms

In this category we would place those branches of mathematics that are based on the same logic (viz. two valued classical) but have different sets of axioms pertaining to the same area of mathematics.

For example:

Euclidean and Non-Euclidean geometries

Cantorian and Non-Cantorian theories

There are several non-Euclidean geometries and non-Cantorian set theories.

Of the latter variety, let us mention set theories:

- without axiom of choice (Agatha Walczak-Typke)
- with the negation of the axiom of choice (Bair Property)
- with dependable/countable choice (DC/CC)
- where continuum hypothesis fails (Web Address: mathoverflow.net/…)
- lacking axiom of extensionality, viz. Weak Set Theory (Scott, Grandy, Bell)
- lacking axiom of foundation, viz. Hyperset theory (Barwise, Moss)

3. Differences in the underlying logics

From the logic-based approach, mathematics has been diverse depending on the use of the underlying logic.

Usually on a base logic a set theory is built. After a stage of development of these diverse set theories attempts are made to construct various other special areas such as number system, analysis, algebras, topological structures and the like.

As examples:

a) Classical set theory based on classical logic

b) Intuitionistic (constructivist) set theory based on intuitionistic logic

c) Quasi set theory based on quantum logic (da Costa, Krause, Dalla Chiara)

Quasi set theory is a way of considering collections of indistinguishable but non- identical objects. The theory has been proposed precisely to provide a mathematical tool for describing collection of objects whose indistinguishability in considered right at the start.

d) Paraconsistent set theory based on paraconsistent logics (Mortensen, Priest, Libert, Weber)

e) Set theory based on many-valued logics (Klaua, Jahn, Gottwald)

Klaua and his co-researchers in Leipzig published "a neglected series of papers" (1965-1970) on many-valued set theory based on Lukasiewicz logic. Klaua got the impulse in the direction of research from Karl Schröter and Karl Menger. In a way these papers may be regarded as generalizations of fuzzy set theory developed independently by Zadeh.

f) Set theories in fuzzy logic (Hajek, Gilder)

- Vopenka and Sochor have been publishing a series of papers centering around ‘mathematics in alternative set theory’ since 1979 some of which had been reported in Prague Topological symposium 1981.

- It would be an interesting project to investigate into the reasons why mathematics based on two-valued logic and Cantorian set theory is projected as the only mathematics in the community.
- Who control?
- Why a graduate student knows Brouwer's fixed point theorems without knowing that Brouwer's main contribution in mathematics is Intuitionism?

4. (other) Differences in the foundations:

- Although all the above aspects are considered in discussions on the foundations of mathematics, there are some other that do not directly fall within the categories 1 – 3 but deserve consideration.

- They are the differences caused from:
  a) Category Theoretic approach;
  b) Many Worlds approach; and
  c) Computer assisted mathematics.

E. B. Davis, "Pluralism in Mathematics", 2004

------, "A Defence of Mathematical Pluralism", 2004.

G. Hellman and J. L. Bell, "Pluralism and the Foundations of Mathematics", (forthcoming).

5. Differences in interpretations

- Due to the above mentioned reasons differences have cropped up regarding
  - the ontology of mathematical entities, and
  - the nature of mathematical truth

Or

- One can say that due to differences in ontology and nature of truth, various approaches have cropped up in mathematics.

6. Differences due to culture, time and history of development of mathematics:

- Differences in mathematical acts may be located in various cultures, in various times and within the history of development of mathematics itself. However, there is not a single sense in which we may understand culture. One may understand existence of culture(s) (i) within the mathematical community; or (ii) across socio-cultural communities.

The question…

- If there exist cognitive differences in ways people across cultures, form their world-view , then will the development of mathematics (across cultures and even across social groups) be the same?
- If the initial development of mathematics is motivated from the needs and functions of a society, then can mathematics grow in isolation, without any impact from the socio-cultural aspects of the community?

Alan Bishop, 'Western Mathematics: the secret of cultural imperialism', 1990, p. 53

"… it is now possible to put forward the thesis that all cultures have generated mathematical ideas, just as all cultures have generated language, religion, morals, customs and kinship systems. Mathematics is now starting to be understood as a pan-cultural phenomenon."

Cognitive Differences:

- Various empirical studies across different kinds of subject groups (inter-culture, intra-culture, inter-society) have revealed the presence of cognitive differences in terms of:
  i. (a) How they categorise the world
     (b) How and to what extent can they perform abstraction from the external world
  ii. Shape (geometric) Perception
  iii. Deduction and Inference
  iv. Reasoning and Problem Solving

Categorisation and Abstraction

1. Patterns of Categorisation in Illiterate Population (Alexander Luria)
2. Patterns of Categorisation amongst Americans and East-Asians (Nisbett, et al)

Studies by Alexander Luria:

- Luria's general purpose was to show the socio-historical roots of all basic cognitive processes; the structure of thought depends upon the structure of the dominant types of activity in different cultures.

- His studies with literate and semi-literate groups within a particular society indicated cognitive differences.

Mode of Categorisation:

Rakmat (39, illiterate peasant):

Shown drawings of: hammer - saw - log - hatchet

Subject: "They are all alike. I think all of them have to be here. See, if you're going to saw, you need a saw, and if you have to split something you need a hatchet. So they are all needed here."

Experimenter: Look, here *you have three adults and one child*. Now clearly the child does not belong in this group.

Subject: "Oh, the boy must stay with the others! All three of them are working, you see, and if they have to keep running to fetch things, they'll never get a job done, but the boy can do the running for them… The boy will learn; that will be better, then they'll be able to work well together."

Experimenter: Look, here you have three wheels and a pair of pliers. Surely the pliers and the wheels aren't alike in any way, are they?

Subject: "No, they all fit together. I know that the pliers do not look like the wheels, but you will need them if you have to tighten something in the wheels."

- Such studied revealed a tendency to categorise objects in a holistic manner, where situations are created in which different objects can co-exist and co-function as a group.

- However, this tendency was not observed among people from the same society exposed to some level of education.

Studies by Nisbett et al.

- Nisbett et al. studied cross-cultural groups (Americans and East-Asians comprising Chinese, Japanese and Koreans). They observed that the following differences between Greek and Chinese cultures regarding:

(a) World-View:

G: Individualistic / Agent Centric / Discrete

C: Holistic / Harmony / Collective Agency

(b) Categorisation:

G: Essential Feature Abstraction necessary for categorising objects and events of the world

C: The concern with abstraction characteristic of ancient Greek philosophy has no counterpart in Chinese philosophy.

(c) Science and Mathematics:

G: Non-tolerance of Contradiction / Logic

C: Tolerance of Contradiction / Dialectism

Mode of Categorisation:

- Based on historical evidence for cognitive differences and Nisbett's theory about the social origins of them, Westerners would:
  - have a greater tendency to categorise objects than would Easterners;
  - find it easier to learn new categories by applying rules about properties to particular cases;
  - make more inductive use of categories, i.e. generalise from particular instances of a category to other instances or to the category as a whole.

Results: Study 1

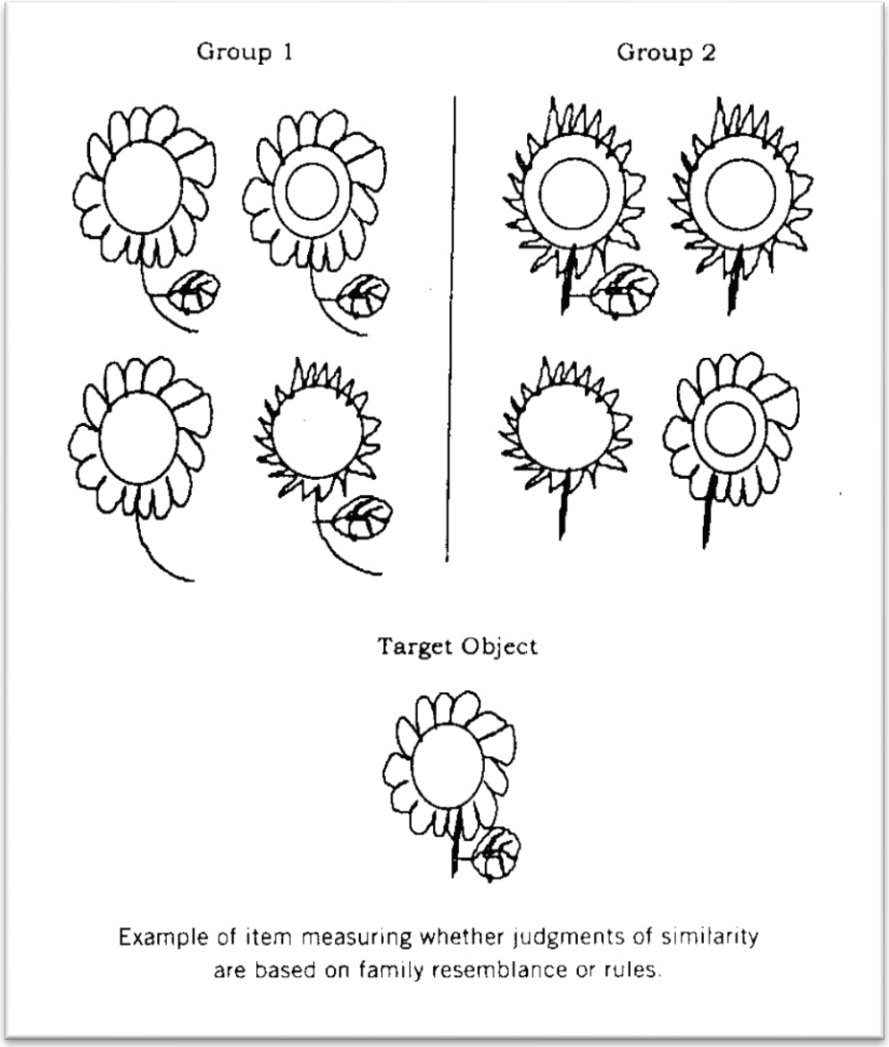
Participants were presented with sets of 3 words and asked them to indicate which 2 of the 3 were most closely related :-

e.g. panda – monkey – banana

American Participants: Displayed a marked preference for grouping on the basis of common category membership. Panda and monkey fit into the animal category.

Chinese Participants: Displayed a marked preference for grouping on the basis of thematic relationships (banana and monkey) and justified their answers in terms of relationships – Monkeys eat bananas.

Study 2:



Example of item measuring whether judgments of similarity are based on family resemblance or rules.

Koreans: The target object was similar to Group 1 (family resemblance).

European Americans: The target object was similar to Group 2. (A rule that allows the object to belong to Group 2, i.e. the object "has a straight stem (as opposed to a curved one)".

"If the natural way of organizing the world for Westerners is to do in terms of categories and the rules that define them, then we might expect that Westerners' perceptions of similarity between objects would be heavily influenced by the degree to which the objects can be categorized by applying a set of rules. But if the categories are less salient to East Asians, then we might expect that their perceptions of similarity would be based more on the family resemblance among objects."

- Richard E. Nisbett (The Geography of Thought: How Asians and Westerners Think Differently… and Why, 2003, p.142)

Deduction and Inference

1. Of illiterate, peasant group of Central Asia (Alexander Luria)
2. Of Kpelle farmers in West Africa (Cole et.al)

The questions…

- Are the logical schemas invariant at different stages of social and historical development?
- Do people have the same form in productive thinking processes in different cultures?
- Are people equally engaged in concrete thinking processes in successive phases of cultural development?
- What exactly is the structure of derivational and inferential processes among people whose life rests upon concrete practical activity?

"A person capable of abstract thought reflects the external world more profoundly and completely and makes conclusions and inferences from perceived phenomena on the basis not only of his personal experience but also of schemes of logical thinking that objectively take shape in a fairly advanced stage of development of cognitive activity."

"The appearance of verbal and logical codes enabling one to abstract the essential features of objects and thus assign these objects to general categories leads to the formation of a more complex logical apparatus. This apparatus permits conclusions to be drawn from given premises without having to resort to immediate graphic-functional experience, and make it possible to acquire new knowledge in a discursive and verbal-logical fashion."

- Luria, Cognitive Development: Its Cultural and Social Foundations, 1976.

Luria:

- Luria (1976) was interested in the effect of literacy on the cognitive capacities of individuals. He experimented with syllogistic problems of the form:

  All A are B

  A certain *x* is an A

  Is this certain x a B? (or a WH-question, such as "What about this certain *x*?)

- The problem that was given to the subjects:

  In the Far North, where there is snow, all bears are white.

  Novaya Zemlya is in the Far North.

  What colours are bear there?

Response:

S: I don't know what colour the bears are there, I never saw them.

E: But what do you think?

S: Once I saw a bear in a museum, that's all.

E: But on the basis of what I said, what colour do you think the bears are there?

S: Either one-coloured or two-coloured … [ponders for a long time]. To judge from the place, they should be white. You say that there is a lot of snow there, but we have never been there!

Subject: Khamark, age: 40, miller from a remote village, illiterate

Results:

Cotton can only grow where it is hot and dry.
In England it is cold and damp.
Can cotton be grown there?

Abdurakhm, age: 37, illiterate, responded:

S: I've only been in Kashgar country; I don't know beyond that.

E: But on the basis of what I said to you, can cotton be grown there?

S: If the land is good, cotton will grow there, but if it is damp and poor, it won't grow. If it's like the Kashgar country, it will grow there too. If the soil is loose, it can grow there too, of course.

Cole at al:

All Kpelle men are rice farmers,

Mr. Smith is not a rice farmer,

Is he a Kpelle man?

S: I don't know the man in person. I have not laid eyes on the man himself.

E: Just think about the statement.

S: If I know him in person, I can answer that question, but since I do not know him in person I cannot answer that question.

Observations:

- There is a reluctance to draw any conclusion about things that are beyond personal experience.
- They do not think that the premises gives information which can be used to answer the question.
- However, people with some level of education did not have the problem of answering the question based upon the information that is given in the premises.

Indian Mathematical Scenario

Some Indian Mathematics:

Aryabhat (476-550)

Value of π : $\dfrac{(100+4)\times 8 + 62000}{20000}$

$= 3.1416$ (approx)

Arithmetic

Solution of diophantine equations of the form $ax - by = c$

Trigonometry

Barahmihir (505-587)

Astronomy/ Astrology

Trigonometry/ Spherical trigonometry

Brahmagupta (628-670)

Considered 0 as a number

He took $0/0 = 0$

Algebraic equations

$Nx^2 \pm C = y^2$ (Pell equation)

Geometry

Trigonometry : $\dfrac{a}{SinA} = \dfrac{b}{SinB} = \dfrac{c}{SinC}$

Astronomy

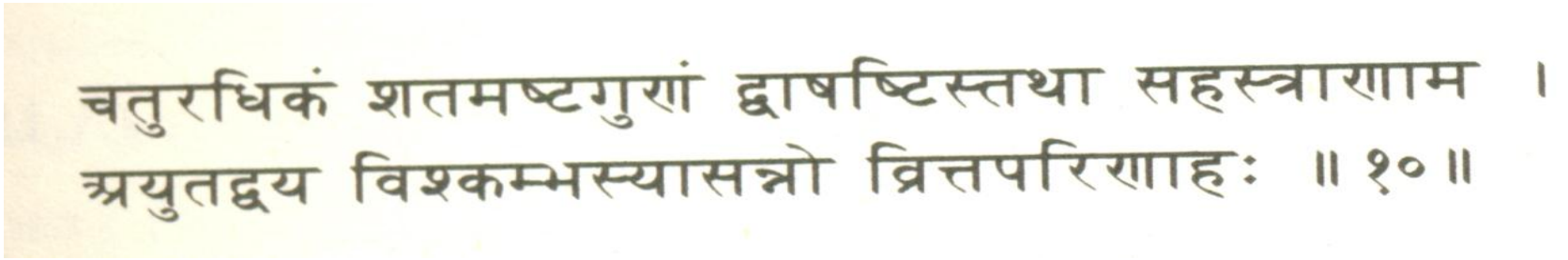
Shridharacharya (870-930)

Quadratic equation $ax^2 + bx = c$

Some markable features of Indian Mathematics

- Written in Sanskrit verse until the first century A.D.

For example:


चतुरधिकं शतमष्टगुणं द्वाषष्टिस्तथा सहस्त्राणाम ।
अयुतद्वय विश्कम्भस्यासन्नो व्रित्तपरिणाहः ॥ १० ॥

Translation:

4 more than 100, multiplied by 8, and added to 62,000: this is the approximate [*asanna*] measure of the circumference of a circle whose diameter is 20,000.

C. K. Raju (2007), *Cultural Foundations of Mathematics: The Nature of Mathematical Proof and the Transmission of the Calculus from India to Europe in the 16th c. CE.*

Some markable features of Indian Mathematics

- Not formalised, algorithmic in nature.

"The methods of calculation were regarded as valuable, not the proofs by themselves – there was no pretence that rationale provided any kind of absolute certainty or necessary truth. Rationale was not valued for its own sake. Hence, rationale was not considered worth recording in many of the terse (*sutra*-style) authoritative texts on mathematics, astronomy, and timekeeping. On the other hand, rationale was not absent, but was taught, as is clear, for example, from the very title *Yuktibhasa*, or in full form, the *Ganita Yuktibhasa,* which means "discourse on rationale in mathematics".

- Raju, p. 72

Some markable features of Indian Mathematics

- Collectives in a nascent form was present.

  "… the Nyaya-Vaisesika system needs to grant ontology status to aggregates (*samudaya*) as distinct from their constituent elements (*samudayin*). … The structure has the aggregate of parts as its support or locus and not any single part. Therefore the Nyaya-Vaisesika theory of wholes and parts can not be made intelligible without admitting in to the system aggregates as legitimate objects (*padarthas*) whose entity-hood though presupposing the entity-hoods of their members, can not be analyzed solely in terms of them."

  B. Pahi (2008), "On Relating The Two Traditions of Logic", *Logic, Navya-Nyaya & Applications.*

- Numbers considered as properties of objects not sets. The same object may obtain different numerocity depending upon the set it belongs to.

What has been presented so far may be summarised as:

- mathematics belongs to the realm of cultural activity of humankind,
- but culture is to be understood in multiple sense, viz.,

    a) various cultures within the mathematics community itself

    b) various mathematical practices that prevailed in different socio-cultural communities

"The direction that mathematics takes in a culture is guided by the cultural needs or attributes – religious, philosophical, agricultural, navigational, industrial as well as mathematical – of the culture. The different directions taken by Greek and Chinese mathematics were determined by the primitive cultural conditions prevailing in these cultures during their prehistoric periods." - Wilder

What has been presented so far may be summarised as:

- mathematical practices in various cultures of category (a) are dominated among other factors by power centres and subjected to homogenization.

  [This might be an explanation of Manin's observation that no major change in methods has taken place in the past 200 years in mathematics.]

- mathematical practices in cultural communities in category (b) have been subjugated by the so called "cultural imperialism."

  [Bishop]

- Since mathematics has no unique "goal" so to say [its arrows are not converging] there is no unique path of its development and hence an attempt for homogenization would be counter-productive.

What has been presented so far may be summarised as:

- Like music or art, mathematics should be given freedom to develop in its own soil (culture) both intra-mathematical and extra-mathematical.

Mihir Kr. Chakraborty, 'Mathematical Rupas and Their Artists', *History of Science, Philosophy and Culture in Indian Civilizationl*, Vol. XIII, Part 7.

What has been presented so far may be summarised as:

- Time is ripe. Technological advancement has been brought to us the opportunity to look beyond simply explaining the past history of mathematics to create the environment in favour of opening up various possibilities that are in the sight.

  "The essence of mathematics is in its freedom." - Cantor

Let a hundred flowers bloom

(together with some weeds)

THANK YOU