Homework Set #1

Capita Selecta: Set Theory
2016/17: 1st Semester; block a
Universiteit van Amsterdam

Homework. There will be six homework sheets; handed in by each student individually. Homework has to contain the name and student ID of the student. If your homework is handwritten, make sure that it is legible. You submit your solutions either by e-mail to `h(dot)nobrega(at)uva(dot)nl` or in person before the Tuesday lecture or by placing them in Hugo's mailbox at the ILLC at Science Park 107.

Final grade. The final grade will be calculated as the average (according to the guidelines of the OER).

Deadline. The first homework is exceptionally due on **Thursday, 15 September 2016** before the lecture.

In all of the exercises, work in a sufficiently strong metatheory, e.g., ZFC plus the existence of set models of all relevant theories needed in the exercise.

1. Let $(M,E) \models \mathsf{ZFC}$. Suppose that $m_0 \in M$ represents the empty set (i.e., $M \models \varphi_\varnothing(m_0)$ where $\varphi_\varnothing(x) := \forall z(\neg z \in x))$ and $m \in M$ is arbitrary. Define precisely by transfinite recursion the *shift function* $\mathbf{shift} : M \to M$ that shifts m_0 to m and all other elements of M accordingly (E.g., if $x \in M$ is the element representing $\{\varnothing\}$, then $\mathbf{shift}(x)$ is the element that represents the singleton whose only element is m, etc.) Let $S := \{\mathbf{shift}(x)\,;\, x \in M\} \subseteq M$. Show that $\mathbf{shift}$ is a structure preserving bijection between (M,E) and (S,E).

   Note. Be careful with your notation: M is a model of the pairing axiom, so for $x,y \in M$ there is some z such that z represents the pair consisting of x and y, i.e., $M \models \forall w(w \in z \leftrightarrow w = x \vee w = y)$; but that does not mean that $\{x,y\} \in M$.

2. Show that the formula $\varphi_\varnothing$ is not preserved under superstructures that are models of ZFC, i.e., there are models $(M,E),(M^*,E^*) \models \mathsf{ZFC}$ with $M \subseteq M^*$ and $E = E^* \cap (M \times M)$ and some $x \in M$ such that $(M,E) \models \varphi_\varnothing(x)$, but $(M^*,E^*) \models \neg\varphi_\varnothing(x)$.

   Hint. Follow the strategy mentioned in the lecture:

   (a) Take an arbitrary model $(M',E') \models \mathsf{ZFC}$ and take a nontrivial shift function $\mathbf{shift}$ as defined in Exercise 1 and form $S := \{\mathbf{shift}(x)\,;\, x \in M'\} \subseteq M'$.

   (b) Take an isomorphic copy $(M,E) \simeq (M',E')$ such that $M \cap M' = \varnothing$. Then by Exercise 1, (S,E') and (M,E) are isomorphic (and $S \cap M = \varnothing$).

   (c) Replace the elements of S appropriately by the elements of M to construct a new model (M^*,E^*). Given an explicit definition of the relation E^*.

   (d) Show that (M,E) is a substructure of (M^*,E^*) and that there is an $x \in M$ such that $(M,E) \models \varphi_\varnothing(x)$, but $(M^*,E^*) \models \neg\varphi_\varnothing(x)$.

3. As usual, the *von Neumann hierarchy* is defined by transfinite recursion: $\mathbf{V}_0 := \varnothing$, $\mathbf{V}_{\alpha+1} := \wp(\mathbf{V}_\alpha)$, and $\mathbf{V}_\lambda := \bigcup_{\alpha\in\lambda} \mathbf{V}_\alpha$ (for λ limit). Show that:

   (a) For every ordinal α, $\mathbf{V}_\alpha$ is a transitive set.

   (b) For $\alpha < \beta$, $\mathbf{V}_\alpha \subseteq \mathbf{V}_\beta$.

   (c) For every ordinal α, $\alpha \in \mathbf{V}_{\alpha+1} \backslash \mathbf{V}_\alpha$.

4. We write

$$\mathrm{ind}(x) := \exists z \varphi_\varnothing(z) \wedge z \in x \wedge \forall w(w \in x \to \exists v(v \in x \wedge \forall y(y \in v \leftrightarrow y \in w \vee y = w)))$$

   for "x is an inductive set". The infinity axiom Inf is then $\exists x(\mathrm{ind}(x))$. Show in detail that $\mathbf{V}_\omega \models \neg\mathrm{Inf}$.

   Hint. Find an appropriate property that all elements of $\mathbf{V}_\omega$ share (prove that they do), but no inductive set in any model of set theory can have.