Homework Set #9

Axiomatische Verzamelingentheorie
2012/13: 2nd Semester
Universiteit van Amsterdam

Please hand in the homework before the start of the Wednesday *werkcollege* (1pm). Late homework will not be accepted. **Takanori and Zhenhao do not accept electronic submissions anymore**. The homework should contain the full names and student ID numbers of all students who contributed. Each homework solution should have at most two names of students. The homework handed in must be the work of the students named on that homework solution. If your homework is handwritten, make sure that it is legible. Also, make sure that you write in complete English sentences.

This homework set is due on **Wednesday 17 April 2013, 1pm**.

1. Show the following algebraic properties of ordinal addition and multiplication:
   (a) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.
   (b) $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.
   (c) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.

2. Remember that an ordinal $\alpha > \omega$ was called an *epsilon-number* if for any $\beta, \gamma \in \alpha$, we have $\beta^\gamma \in \alpha$. Remember that for any fixed $\xi \notin \{0, 1\}$, the operation $E_\xi : \alpha \mapsto \xi^\alpha$ has arbitrarily large fixed points. Show that the following are equivalent:
   (a) α is an epsilon-number, and
   (b) for all $\xi < \alpha$, if $\xi \notin \{0, 1\}$, then α is a fixed point of E_ξ.

   Define $e_0 := \omega$, $e_{n+1} := \omega^{e_n}$, and $\varepsilon_0 := \bigcup\{e_n\,;\, n \in \omega\}$. Show that ε_0 is the least epsilon-number.

3. Show the following algebraic properties of ordinal exponentiation (assume $\alpha \neq 0$ since the function $x \mapsto 0^x$ has some weird properties):
   (a) $\alpha^\beta \cdot \alpha^\gamma = \alpha^{\beta+\gamma}$.
   (b) $(\alpha^\beta)^\gamma = \alpha^{(\beta\cdot\gamma)}$.

   Define $\alpha_0 := \omega$, $\alpha_{n+1} := (\alpha_n)^\omega$, and $\alpha := \bigcup\{\alpha_n\,;\, n \in \omega\}$. Show that $\alpha \neq \varepsilon_0$. (In fact, there is an n such that $\alpha = e_n$ in the notation of exercise 2. Find n.)