# Homework Set #8

**Axiomatische Verzamelingentheorie**
*2012/13: 2nd Semester*
Universiteit van Amsterdam

Please hand in the homework before the start of the Wednesday *werkcollege* (1pm). Late homework will not be accepted. **Takanori and Zhenhao do not accept electronic submissions anymore**. The homework should contain the full names and student ID numbers of all students who contributed. Each homework solution should have at most two names of students. The homework handed in must be the work of the students named on that homework solution. If your homework is handwritten, make sure that it is legible. Also, make sure that you write in complete English sentences.

This homework set is due on **Wednesday 10 April 2013, 1pm**.

1. Remember that we had two versions of the Axiom of Foundation: The *axiom of foundation* states
$$\forall x\,(x \neq \varnothing \rightarrow \exists y\,(y \in x \wedge y \cap x = \varnothing))$$
and the *axiom scheme of foundation* states for every formula $\varphi$ (possibly with parameters) the following is true:
$$\exists z(\varphi(z)) \rightarrow \exists y\,(\varphi(y) \wedge \forall w\,(w \in y \rightarrow \neg\varphi(w)))\,.$$
Clearly, the axiom scheme implies the axiom (just take $\varphi(z) :\Longleftrightarrow z \in x$). Show that in the context of all other axioms of set theory, the axiom implies the axiom scheme.

   (**Hint.** Consider the transitive closure from Exercise 5. on homework set #7.)

2. Prove the *Fundamental Theorem of Wellorders* stating that for any two wellorders $(X, R)$ and $(X^*, R^*)$ one of the following three statements holds:
   (a) $(X, R)$ and $(X^*, R^*)$ are isomorphic,
   (b) $(X, R)$ is isomorphic to a proper initial segment of $(X^*, R^*)$, or
   (c) $(X^*, R^*)$ is isomorphic to a proper initial segment of $(X, R)$.

   Follow the proof sketch we gave in class and do not use the conclusion of Exercise 3.: define by recursion a (possibly partial) map $\pi : X \rightarrow X^*$ that picks the $R^*$-least element of $X^*$ that hasn't been picked before; deal with the case that we run out of elements of $X^*$ before we run out of elements of $X$; consider $\mathrm{dom}(\pi)$ and $\mathrm{ran}(\pi)$ and prove that $\pi$ is an isomorphism between the suborders on these two sets; prove that at least one of these two sets is the entire set; conclude that the theorem is true.

3. Consider the total operation defined by
$$\Phi(x, y) :\Longleftrightarrow x \text{ is a function and } y = \mathrm{ran}(x) \text{ or } x \text{ is not a function and } x = y.$$
Let $(X, R)$ be any wellorder and define a function $F$ with $\mathrm{dom}(F) = X$ by recursion via the operation defined by $\Phi$. Show that the range of $F$ is an ordinal and that $F$ is an isomorphism between $(X, R)$ and $(\mathrm{ran}(F), \in)$. Use this to give a simpler proof of the *Fundamental Theorem of Wellorders* from Exercise 2.