Homework Set #6

Axiomatische Verzamelingentheorie
2012/13: 2nd Semester
Universiteit van Amsterdam

The homework can be handed it on paper in the Friday lectures. Please hand in the homework before the start of the Friday lecture (1pm). Late homework will not be accepted. **Takanori and Zhenhao do not accept electronic submissions anymore**.

The homework should contain the full names and student ID numbers of all students who contributed. Each homework solution should have at most two names of students. The homework handed in must be the work of the students named on that homework solution. If your homework is handwritten, make sure that it is legible. Also, make sure that you write in complete English sentences.

This homework set is due on **Friday 22 March 2013, 1pm**.

1. If $(X, <)$ is a strict linear order, then a function $s : X \to X$ is called a *successor function* if for all $x \in X$, we have $x < s(x)$ and there is no x' such that $x < x' < s(x)$. Define the following relation $\prec$ on $\mathbb{N}$:
$$n \prec m \iff \begin{cases} & \exists k \exists \ell (2k = n \wedge 2\ell = m \wedge k < \ell) \\ \vee & \exists k \exists \ell (2k+1 = n \wedge 2\ell + 1 = m \wedge k < \ell) \\ \vee & \exists k \exists \ell (2k = n \wedge 2\ell + 1 = m) \text{ and} \end{cases}$$
the function $S : \mathbb{N} \to \mathbb{N}$ be defined by $S(n) := n + 2$.
   (a) Prove that $(\mathbb{N}, \prec)$ is a wellorder.
   (b) Prove that S is a successor function for $(\mathbb{N}, \prec)$.
   (c) Show that there is a set $A \subsetneq \mathbb{N}$ such that $0 \in A$ and if $x \in A$, then $S(x) \in A$.

2. Let $(X, <)$ be a wellorder and $s : X \to X$ a successor function (as in exercise 1.). We call the elements of $\mathrm{ran}(s)$ *successors* and the elements of $X \backslash \mathrm{ran}(s)$ *non-successors*. For example, the successor function S in exercise 1. has exactly two non-successors, viz. 0 and 1.

   Is it possible to define a binary relation $\prec$ on $\mathbb{N}$ and a function $S : \mathbb{N} \to \mathbb{N}$ such that $(\mathbb{N}, \prec)$ is a wellorder, S is a successor function $(\mathbb{N}, \prec)$, and there are infinitely many non-successors for S?

   If it's possible, give such a definition and prove that it has these properties; if not, prove that this cannot be done.

3. Let $(X, <)$ be a wellorder. If $x \in X$, we write $\mathrm{IS}(x) := \{x' \in X\,;\, x' < x\}$ for the initial segment of elements below x. The set $\mathrm{IS}(x)$ inherits a wellorder from $<$ (since subsets of wellorders are wellorders). Let $\mathcal{F}(X)$ be the set of all functions defined on an initial segment of X with range X, i.e.,
$$f \in \mathcal{F} \iff \exists x \in X(\mathrm{dom}(f) = \mathrm{IS}(x) \wedge \mathrm{ran}(f) \subseteq X).$$
   Prove that the following local version of the recursion theorem holds (without using the Recursion Theorem that we'll prove on 19 March in the lecture):

   If $r : \mathcal{F} \to X$, then there is a function $F : X \to X$ such that for all $x \in X$, we have
$$F(x) = r(F \restriction \mathrm{IS}(x)).$$