Homework Set #4

Axiomatische Verzamelingentheorie
2012/13: 2nd Semester
Universiteit van Amsterdam

The homework can be handed it electronically to Takanori Hida (`t.hida@uva.nl`) or on paper in the Friday lectures. Please hand in the homework before the start of the Friday lecture (1pm). Late homework will not be accepted.

The homework should contain the full names and student ID numbers of all students who contributed. Each homework solution should have at most two names of students. The homework handed in must be the work of the students named on that homework solution. If your homework is handwritten, make sure that it is legible. Also, make sure that you write in complete English sentences.

This homework set is due on **Friday 8 March 2013, 1pm**.

1. Consider the notion of the Zermelo inductive set from homework set #3 where you proved that if there is a Zermelo inductive set, then there is a least Zermelo inductive set. Assume in this exercise that there is a least Zermelo inductive set called $\mathbb{N}_{\text{Zermelo}}$. We call its elements the *Zermelo natural numbers.*

   (a) Show that for every Zermelo natural number x, we have $x \notin x$.

   (b) Show that every Zermelo natural number has at most one element.

   (c) We can define a structure $\mathcal{N} := (\mathbb{N}_{\text{Zermelo}}, \varnothing, S^*)$ where $S^*(x) := \{x\}$. Show that $\mathcal{N}$ is a Peano structure.

2. We defined the following formula

   $$\Phi_{\text{plus}}(x, y, z) :\iff \exists a \exists b \exists f \exists g (a \cap b = \varnothing \wedge a \cup b = z \wedge \text{Bij}(f, x, a) \wedge \text{Bij}(g, v, b))$$

   where $\text{Bij}(h, v, w)$ is an abbreviation for the formula expressing "h is a bijection from v to w". Show that this formula defines a total operation on $\mathbb{N}$, i.e., that for every $x, y \in \mathbb{N}$ there is a unique $z \in \mathbb{N}$ such that $\Phi(x, y, z)$ holds.

   (**Hint.** For the uniqueness, you need to show that for every natural number n, there can be no bijection between n and some $k < n$. For the existence, fix y and prove the existence of z for that fixed y by induction.)

3. Prove that (Ex)+(Ext)+(Pow)+(Aus)+(Repl) proves (Pair).