Homework Set #3

Axiomatische Verzamelingentheorie
2012/13: 2nd Semester
Universiteit van Amsterdam

The homework can be handed it electronically to Takanori Hida (`t.hida@uva.nl`) or on paper in the Friday lectures. Please hand in the homework before the start of the Friday lecture (1pm). Late homework will not be accepted.

The homework should contain the full names and student ID numbers of all students who contributed. Each homework solution should have at most two names of students. The homework handed in must be the work of the students named on that homework solution. If your homework is handwritten, make sure that it is legible. Also, make sure that you write in complete English sentences.

This homework set is due on **Friday 29 February 2013, 1pm**.

1. We define the *singleton axiom*, a weakening of the axiom (Pair):

$$\forall x \exists s \forall z (z \in s \leftrightarrow z = x) \tag{Sing}$$

   Clearly, (Pair) implies (Sing).

   [**Remark.** Note that our proof that models of the pairing axiom cannot be finite actually only used the singleton axiom, so in the presence of (Ext)+(Aus), models of (Sing) cannot be finite.]

   Construct a model (necessarily infinite) in which (Ex), (Ext), (Aus), (Sing), and (Un) hold, but (Pair) does not. (You have to *prove* that your model satisfies all of the axioms you claim it satisfies.)

2. We called a set I *inductive* if $\varnothing \in I$ and if $x \in I$, then $x \cup \{x\} \in I$. In analogy, let us call a set *Zermelo inductive* if $\varnothing \in I$ and if $x \in I$, then $\{x\} \in I$. Show that if there is a Zermelo inductive set, then there is a least Zermelo inductive set which we shall call $\mathbb{N}_{\text{Zermelo}}$. Show that $\bigcup \mathbb{N}_{\text{Zermelo}} = \mathbb{N}_{\text{Zermelo}}$.

3. Consider two disjoint copies of the model **HF** of the hereditarily finite sets constructed in class and call them $\mathcal{H}_0 = (H_0, E_0)$ and $\mathcal{H}_1 = (H_1, E_1)$. Each of the two models has a unique bottom vertex (the empty set) which we shall call e_0 and e_1, respectively. Let e be a vertex that doesn't occur in either H_0 or H_1. We construct a new model $\mathcal{M} = (M, E)$ as follows:

   - $M := \{e\} \cup H_0 \backslash \{e_0\} \cup H_1 \backslash \{e_1\}$;
   - if $x \in H_i$, then eEx iff $e_i E_i x$;
   - for any $x, y \in H_i$, we let xEy iff xE_iy;
   - if $x \in H_0$ and $y \in H_1$, then xEy and yEx do **not** hold.

   Show that $\mathcal{M}$ satisfies (Ex), (Un), (Pow), and (Aus), but not (Ext), (Pair), and (BinUn).

   Extra question for extra credit. Can you do the same while also having (Ext), i.e., a model of (Ex), (Ext), (Un), (Pow), and (Aus), but not (Pair) and (BinUn)?