Homework Set #2

Axiomatische Verzamelingentheorie
2012/13: 2nd Semester
Universiteit van Amsterdam

The homework can be handed it electronically to Takanori Hida (`t.hida@uva.nl`) or on paper in the Friday lectures. Please hand in the homework before the start of the Friday lecture (1pm). Late homework will not be accepted.

The homework should contain the full names and student ID numbers of all students who contributed. Each homework solution should have at most two names of students. The homework handed in must be the work of the students named on that homework solution. If your homework is handwritten, make sure that it is legible. Also, make sure that you write in complete English sentences.

This homework set is due on **Friday 22 February 2013, 1pm**.

1. In class, we proved that (Ex)+(Ext)+(Aus)+(Pair) cannot hold in finite models. Can you show the same for the weaker theory (Ex)+(Ext)+(Pair)?

   If yes, prove it. If no, produce a finite countermodel and show that the theory holds in that model.

2. Prove that the Kuratowski pair $(x, y) := \{\{x\}, \{x, y\}\}$ satisfies the following:

$$(x, y) = (x', y') \iff x = x' \text{ and } y = y'.$$

3. We define a sequence of models (M_i, E_i) by recursion:

   - $M_0 := \{z\}$ for some fixed vertex z and $E_0 := \varnothing$. So, $\mathcal{M}_0 = (M_0, E_0)$ is our model $\mathcal{M}_0$ from the lecture.
   - If $\mathcal{M}_i = (M_i, E_i)$ is defined, consider every possible pair of elements $x, y \in M_i$ and follow the following procedure: Check whether there is a $p \in M_i$ such that

$$\forall z(z \in p \leftrightarrow (z = x \vee z = y)).$$

     If yes, then let $p_{xy} := p_{yx} := p$ for this uniquely defined p.

     If no, then take a new vertex $p_{xy} = p_{yx}$ that hasn't been used before.

     Let

$$M_{i+1} := M_i \cup \{p_{xy}\,;\, x, y \in M_i\}$$

     and

$$E_{i+1} := E_i \cup \{(x, p_{xy})\,;\, x, y \in M_i\} \cup \{(y, p_{xy})\,;\, x, y \in M_i\}.$$

   In the end, let $M := \bigcup_{n\in\mathbb{N}} M_n$, $E := \bigcup_{n\in\mathbb{N}} E_n$, and $\mathcal{M} := (M, E)$. Show that (Ex), (Ext), (Aus), and (Pair) hold in $\mathcal{M}$.

   [**Hint.** For (Aus), first show by induction that every $\mathcal{M}$-set has at most two $\mathcal{M}$-elements and use this to bound the number of subclasses you need to check for the validity of (Aus).]