Homework Set #11

Axiomatische Verzamelingentheorie
2012/13: 2nd Semester
Universiteit van Amsterdam

Please hand in the homework before the start of the Wednesday *werkcollege* (1pm). Late homework will not be accepted. **Takanori and Zhenhao do not accept electronic submissions anymore**. The homework should contain the full names and student ID numbers of all students who contributed. Each homework solution should have at most two names of students. The homework handed in must be the work of the students named on that homework solution. If your homework is handwritten, make sure that it is legible. Also, make sure that you write in complete English sentences.

This homework set is due on **Wednesday 8 May 2013, 1pm**.

Please keep track of the time that you spend on this homework and write it on top of your answer sheet. Of course, this information will not be used in a personalized way, but only to get an estimate for the average weekly work load.

In the following, AC_Z is the statement "for every $X \preceq Z$, if X consists only of non-empty sets, then there is a choice function for X" and AC is $\forall Z(\mathsf{AC}_Z)$, as in the lecture.

1. Let X be a set of non-empty sets. We call C a *choice set for* X if for all $x \in X$, we have that $|C \cap x| = 1$. Give an example of a set X consisting of three non-empty sets such that there is no choice set for X.

   Consider the following variant of the axiom of choice:

   $$\text{Every set } X \text{ of non-empty, pairwise disjoint sets has a choice set.} \qquad (\mathsf{AC}')$$

   Show that AC and AC' are equivalent over ZF.

2. Work in ZF. In the lecture, we called a cardinal κ *regular* if for every subset $X \subseteq \kappa$ with $|X| < \kappa$ (Why can we use the $|\cdot|$ notation here without assuming that AC holds?) we have that $\bigcup X < \kappa$. Let α be any ordinal. Show that $\mathsf{AC}_{\aleph_\alpha}$ implies that $\aleph_{\alpha+1}$ is regular.

3. Work in ZF. We called a set X *Dedekind-infinite* if there is a proper subset $A \subsetneqq X$ such that $A \sim X$. A set is *Dedekind-finite* if it is not Dedekind-infinite.

   (a) Show that every Dedekind-infinite set is infinite.
   (b) Show that a set D is Dedekind-infinite if and only if there is a function $f : D \to D$ that is injective but not surjective.
   (c) Show that a set D is Dedekind-infinite if and only if $\omega \preceq D$.
   (d) Show that AC implies that every infinite set is Dedekind-infinite.
   (e) Assume that there is a set D that is infinite, but Dedekind-finite. Show that D and $\mathbb{N}$ are incomparable in the $\preceq$ order (i.e., $D \not\preceq \mathbb{N}$ and $\mathbb{N} \not\preceq D$).

Extra credit exercise. A set A is called amorphous if it is infinite and for every subset $X \subseteq A$, either X or $A \backslash X$ is finite. (Of course, amorphous sets are Dedekind-finite, and so AC implies that they do not exist.) A set D is called *dually Dedekind-infinite* if there is a function $f : D \to D$ that is surjective but not injective. (Compare Exercise (3b).) If X is any set, we denote by $\mathrm{Seq}(X) := \{s\,;\, \mathrm{dom}(s) \in \mathbb{N}, \mathrm{ran}(s) \subseteq X, \text{and } s \text{ is injective}\}$ the set of finite non-repeating sequences of elements of X.

Assume that there is an amorphous set A. Show that $\mathrm{Seq}(A)$ is dually Dedekind-infinite, but not Dedekind-infinite.