UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Core Logic

2007/2008; 1st Semester
dr Benedikt Löwe

Homework Set # 12 *Deadline:* December 5th, 2007

Exercise 40 (7 points).

Let PA be the first-order axiom system of Peano Arithmetic. Assume that PA is consistent.

(1) Show that there is a model $\mathfrak{M}$ of $\mathsf{PA} + \neg\mathrm{Cons}(\mathsf{PA})$ (1 point).
(2) Give an example of a sentence that is true in $\mathfrak{M}$ but not true in the metatheory (1 point).
(3) Consider the following symmetric version of Gödel's Second Incompleteness Theorem SymG2:
    If T is a consistent recursively axiomatized theory such that $\mathsf{PA} \subseteq T$, then the theories $T + \mathrm{Cons}(T)$ and $T + \neg\mathrm{Cons}(T)$ are consistent as well.
    Give a counterexample to SymG2 (5 points).

Exercise 41 (7 points).

Read the paper

> Richard **Zach**, The practice of finitism: Epsilon calculus and consistency proofs in Hilbert's Program, **Synthese** 137 (2003), p.211-259

and answer the following questions:

(1) Is the following statement true or false? "In his lecture course in *Sommersemester 1920*, Hilbert tried to axiomatize all of mathematics based on Frege's second-order logic from the *Grundlagen*." (1 point)
(2) According to Zach, in which semester did Hilbert use the ε-operator for the first time in his lecture course? (1 point)
(3) In the lectures from *Wintersemester 1921/22*, Hilbert uses operators τ and α. What is the relationship between these and the ε-operator? (2 points)
(4) As opposed to lectures of the earlier semesters, in his lectures of the *Wintersemester 1922/23*, Hilbert does not give axioms for addition and multiplication before the introduction of primitive recursive definitions. What (according to Zach) is the reason for this? (1 point)
(5) There is a major shift in the meaning of the ε-operator between Hilbert's 1923 paper and Ackermann's PhD dissertation (1924). What is it? (1 point)
(6) Who wrote to Hilbert in a letter in 1933 that one of the Hilbert-style consistency proofs "does not seem to harmonize with the work of Gödel"? (1 point)

Exercise 42 (4 points).

Let $2^{\mathbb{N}}$ be the set of all infinite 0-1 sequences. For $x \in 2^{\mathbb{N}}$, we define $\widehat{x}(n) := 1 - x(n)$. We call $\mathcal{C} \subseteq 2^{\mathbb{N}}$ a **symmetric class** if for every $x \in \mathcal{C}$, we also have $\widehat{x} \in \mathcal{C}$. A function $F : \mathbb{N} \to \mathcal{C}$ is called a $\mathcal{C}$-good parametrization if the sequence $\langle F(n)(n)\,;\, n \in \mathbb{N}\rangle$ is an element of $\mathcal{C}$ and F is a surjection. Show that no symmetric class $\mathcal{C}$ can have a $\mathcal{C}$-good parametrization (3 points).

Exercise 43 (4 points).

The ordinal ω_1^{CK} is sometimes called "the least admissible ordinal" and has an equivalent description in terms of an axiom system called "Kripke-Platek Set Theory" KP. Give a precise definition of ω_1^{CK} (2 points) and of KP (2 points) and give a brief (two to four sentences) description of the connection between the ordinal and KP (2 points).