UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Axiomatische Verzamelingentheorie

2005/2006; 2nd Semester
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Homework Set # 7 *Deadline:* April 6th, 2006 (**TWO WEEKS!**)

Important Note. Unless otherwise mentioned, please work in the theory ZF for this homework set, *i.e.*, do not use the axiom of choice. The main goal of these exercises is to make you more sensitive to what arguments need the axiom of choice and which ones don't.
This will require you to be very precise in constructions of objects like injections and surjections (in Exercises 20, 21 and 22). They have to be either defined by means of a single formula or recursively defined by an explicit use of the (transfinite) recursion theorem. Uses of the axiom of choice often hide in the phrases "and so on" or the ellipsis "...". Therefore do **not** use them!

Exercise 19 ("Vitali's Theorem"; total of nine points).

Let $\mathbb{R}$ be the set of real numbers and let $\mathbb{R}^* := \{x \in \mathbb{R}\,;\, x \geq 0\} \cup \{\infty\}$ where ∞ is some additional symbol. We define an ordering on $\mathbb{R}^*$ extending the usual order of $\mathbb{R}$ by $x \leq \infty$ for all x. We extend the operation $+$ on $\mathbb{R}$ to $\mathbb{R}^*$ by $\infty + \infty = \infty + x = x + \infty = \infty$.
A function $\mu : \wp(\mathbb{R}) \to \mathbb{R}^*$ is called a **measure** if

- $\mu(\varnothing) = 0$,
- for $A \subseteq B$, we have $\mu(A) \leq \mu(B)$,
- for any countable family $\{A_c\,;\, c \in C\}$ of sets of reals that are pairwise disjoint (*i.e.*, if $c \neq d$, then $A_c \cap A_d = \varnothing$), we have
$$\mu(\bigcup_{c\in C} A_c) = \sum_{c\in C} \mu(A_c).$$

Give an example of a measure (1 point).

Given a measure μ, we say that a set $X \subseteq \mathbb{R}$ **has finite measure** if $0 < \mu(X) \neq \infty$. We call a measure **nontrivial** if there is a set that has finite measure.
Give an example of a nontrivial measure (1 point).

A measure μ is called **continuous** if for all $x \in \mathbb{R}$, we have $\mu(\{x\}) = 0$. For $x \in \mathbb{R}$ and $A \subseteq \mathbb{R}$, we let $x + A := \{x + a\,;\, a \in A\}$. We call a measure μ **translation-invariant** if for all A and x, we have $\mu(x + A) = \mu(A)$.

Using the Axiom of Choice, prove that there is no translation-invariant continuous non-trivial measure. Use the following steps:

STEP 1. We define an equivalence relation $\equiv_{\mathrm{V}}$ on $\mathbb{R}$ by $x \equiv_{\mathrm{V}} y :\iff x - y \in \mathbb{Q}$. It is called the **Vitali equivalence relation**. Prove that it is an equivalence relation (1 point).
If $\equiv$ is an equivalence relation on a set S, we call $T \subseteq S$ a **transversal for** $\equiv$ if T has exactly one element from each $\equiv$-equivalence class.
Prove (using AC) that there is a transversal for every equivalence relation (4 points).

STEP 2. Assume towards a contradiction that μ is a translation-invariant continuous non-trivial measure and that X is a set that has finite measure. Let T be a transversal for $\equiv_{\mathrm{V}}$ that you get from *STEP 1*.
Prove that $\mu(X \cap T)$ cannot be 0 (1 point). Prove that $\mu(X \cap T)$ cannot be positive. (1 point). Therefore $\mu(X \cap T)$ cannot be defined. Contradiction!

Exercise 20 (total of ten points).

Let $L := \{R \subseteq \omega \times \omega\,;\, R \text{ is a total order}\}$. Define an equivalence relation $\equiv$ on L by $R \equiv R'$ if and only if $\langle \omega, R\rangle$ and $\langle \omega, R'\rangle$ are order-isomorphic. Let $L/\!\equiv$ be the set of $\equiv$-equivalence classes of L. Let ω^ω be the set of functions from ω to ω. Prove that there is an injection from ω^ω into $L/\!\equiv$ (10 points).

(In other words: "There are at least as many order types of total orders as there are real numbers.")

(**Hint.** Using the operation $\oplus$ on tosets as defined in Exercise 8 (on HW Set #3), consider $\mathbb{Z} \oplus \mathbf{1} \oplus \mathbb{Z}$ and $\mathbb{Z} \oplus \mathbf{2} \oplus \mathbb{Z}$, and show that they are not order-isomorphic. Generalize this.)

Exercise 21 (total of fourteen points).

A set D is called **infinite** if there is no natural number n such that there is a bijection between D and n. It is called **Dedekind-infinite** if there is a proper subset $S \subsetneq D$ with a bijection between S and D.

Show that every Dedekind-infinite set is infinite (2 points).

Show that a set D is Dedekind-infinite if and only if there is a subset $N \subseteq D$ such that there is a bijection between N and ω (6 points).

Use AC to show that every infinite set is Dedekind-infinite (6 points).

Exercise 22 (total of eleven points).

Prove that the function $\ulcorner\cdot,\cdot\urcorner : \omega \times \omega \to \omega$ defined by

$$\ulcorner i, j \urcorner := \frac{i^2 + j^2 + 2ij + 3i + j}{2}$$

is a bijection (3 points).

Let 2^ω be the set of functions from ω to $2 = \{0, 1\}$ (*i.e.*, the set of infinite binary sequences) and $(2^\omega)^\omega$ the set of functions from ω to 2^ω. Prove that there is a bijection between 2^ω and $(2^\omega)^\omega$ (1 point).

For $x \in 2^\omega$, define $R_x \subseteq \omega \times \omega$ by

$$i\, R_x\, j :\iff x(\ulcorner i, j\urcorner) = 1.$$

Define

$$F(x) := \begin{cases} \mathrm{o.t.}(\langle \omega, R_x\rangle) & \text{if } R_x \text{ is a wellorder,} \\ \varnothing & \text{otherwise.} \end{cases}$$

(Here $\mathrm{o.t.}$ is the order type of a woset defined in Exercise 9 on HW set #3.)
Let Ω be the class of countable ordinals. Prove that F is a surjection from 2^ω onto Ω (4 points).
Note that we didn't know that Ω is a set – why do we know that now? (1 point)
Use AC to prove that there is an injection from Ω into 2^ω (2 points). (**Hint.** Exercise 5 on HW Set #2.)