UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

# Axiomatische Verzamelingentheorie

**2005/2006; 2nd Semester**
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**Homework Set # 10** *Deadline:* April 27th, 2006

**Exercise 26** (total of ten points).

If $\alpha$ is an ordinal, then sequences $\langle\varepsilon_0, ..., \varepsilon_n\rangle$ and $\langle k_0, ..., k_n\rangle$ with

$$\varepsilon_n > \varepsilon_{n-1} > ... > \varepsilon_1 > \varepsilon_0,$$
$$0 < k_i < \omega, \text{ and}$$
$$\alpha = \omega^{\varepsilon_n} \cdot k_n + \omega^{\varepsilon_{n-1}} \cdot k_{n-1} + ... + \omega^{\varepsilon_0} \cdot k_0$$

is its **Cantor Normal Form to the base** $\omega$ (CNF). Prove that the Cantor Normal Form is unique, *i.e.*, if both $\langle\varepsilon_0, ..., \varepsilon_n\rangle$, $\langle k_0, ..., k_n\rangle$ and $\langle\varepsilon_0^*, ..., \varepsilon_n^*\rangle$, $\langle k_0^*, ..., k_n^*\rangle$ have the above properties, then $\varepsilon_i = \varepsilon_i^*$ and $k_i = k_i^*$ for all $i$ (6 points).

Compute the CNF to the base $\omega$ of $\omega^\omega \cdot \omega^{\omega^{\omega+1}}$ (2 points) and $\omega \cdot 7 + (\omega + 1)^\omega$ (2 points).

**Exercise 27** (total of nine points).

Let $\alpha$, $\beta$ and $\gamma$ be ordinals. Prove that

(1) $\alpha^\gamma \cdot \alpha^\beta = \alpha^{\gamma+\beta}$ (3 points),
(2) $\alpha^\gamma \cdot \beta^\gamma = (\alpha \cdot \beta)^\gamma$ (3 points),
(3) $(\alpha^\gamma)^\beta = \alpha^{\gamma\cdot\beta}$ (3 points).

**Exercise 28** (total of seven points).

As defined in the lecture, an ordinal number $\alpha$ is called an **$\varepsilon$-number** if for all $\xi, \eta < \alpha$, we have $\xi^\eta < \alpha$. Clearly, $\omega$ is an $\varepsilon$-number.

Define the following two numbers $\alpha$ and $\beta$: $\alpha_0 := \omega$, $\alpha_{n+1} := \omega^{\alpha_n}$, $\alpha := \bigcup\{\alpha_n\,;\, n \in \omega\}$ and $\beta_0 := \omega$, $\beta_{n+1} := {\beta_n}^\omega$, $\beta := \bigcup\{\beta_n\,;\, n \in \omega\}$. Show that $\beta < \alpha$ (3 points). Show that $\alpha$ is the least $\varepsilon$-number that is bigger than $\omega$ (4 points). (Clearly, this implies that $\beta$ is not an $\varepsilon$-number.)