Modal Propositional Logic.

- **Propositional Logic:** $\mathbf{Prop}$. Propositional variables p_i, $\wedge$, $\vee$, $\neg$, $\rightarrow$.
- **Modal Logic.** $\mathbf{Prop}+$ $\Box$, $\Diamond$.
- **First-order logic.** $\mathbf{Prop}+$ $\forall$, $\exists$, function symbols $\dot{\mathrm{f}}$, relation symbols $\dot{\mathrm{R}}$.

$$\mathbf{Prop} \quad \subseteq \quad \mathbf{Mod} \quad \subseteq \quad \mathbf{FOL}$$

Standard
Translation

The standard translation (1).

Let $\dot{\mathrm{P}}_i$ be a unary relation symbol and $\dot{\mathrm{R}}$ a binary relation symbol.

We translate $\mathbf{Mod}$ into $\mathcal{L} = \{\dot{\mathrm{P}}_i, \dot{\mathrm{R}}\,;\, i \in \mathbb{N}\}$.

For a variable x, we define ST_x recursively:

$$
\begin{array}{rcl}
\mathrm{ST}_x(\mathrm{p}_i) & := & \dot{\mathrm{P}}_i(x) \\
\mathrm{ST}_x(\neg\varphi) & := & \neg\mathrm{ST}_x(\varphi) \\
\mathrm{ST}_x(\varphi \vee \psi) & := & \mathrm{ST}_x(\varphi) \vee \mathrm{ST}_x(\psi) \\
\mathrm{ST}_x(\Diamond\varphi) & := & \exists y \left(\dot{\mathrm{R}}(x, y) \wedge \mathrm{ST}_y(\varphi)\right)
\end{array}
$$

The standard translation (2).

If $\langle M, R, V\rangle$ is a Kripke model, let $P_i := V(\mathrm{p}_i)$. If P_i is a unary relation on M, let $V(\mathrm{p}_i) := P_i$.

Theorem.

$$\langle M, R, V\rangle \models \varphi \quad \leftrightarrow \quad \langle M, P_i, R\,;\, i \in \mathbb{N}\rangle \models \forall x\, \mathrm{ST}_x(\varphi)$$

Corollary. Modal logic satisfies the compactness theorem.

Proof. Let Φ be a set of modal sentences such that every fi nite set has a model. Look at $\Phi^* := \{\forall x\, \mathrm{ST}_x(\varphi)\,;\, \varphi \in \Phi\}$. By the theorem, every fi nite subset of Φ^* has a model. By compactness for fi rst-order logic, Φ^* has a model. But then Φ has a model. q.e.d.

Bisimulations.

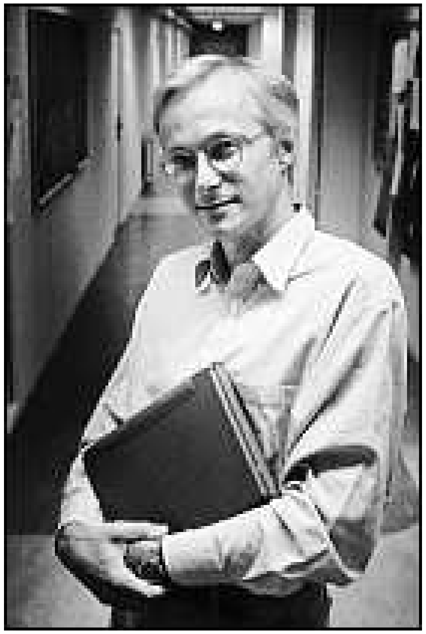
If $\langle M, R, V\rangle$ and $\langle M^*, R^*, V^*\rangle$ are Kripke models, then a relation $Z \subseteq M \times N$ is a bisimulation if

- If xZx^*, then $x \in V(\mathrm{p}_i)$ if and only if $x^* \in V(\mathrm{p}_i)$.
- If xZx^* and xRy, then there is some y^* such that $x^*R^*y^*$ and yZy^*.
- If xZx^* and $x^*R^*y^*$, then there is some y such that xRy and yZy^*.

A formula $\varphi(v)$ is called invariant under bisimulations if for all Kripke models $\mathbf{M}$ and $\mathbf{N}$, all $x \in M$ and $y \in N$, and all bisimulations Z such that xZy, we have

$$\mathbf{M} \models \varphi(x) \leftrightarrow \mathbf{N} \models \varphi(y).$$

van Benthem.


Johan van Benthem

Theorem (van Benthem; 1976). A formula in one free variable v is invariant under bisimulations if and only if it is equivalent to $\mathrm{ST}_v(\psi)$ for some modal formula ψ.

Modal Logic is the bisimulation-invariant fragment of first-order logic.

Decidability.

Theorem (Harrop; 1958). Every finitely axiomatizable modal logic with the finite model property is decidable.

Theorem. $\mathbf{T}$, $\mathbf{S4}$ and $\mathbf{S5}$ are decidable.

Intuitionistic Logic (1).

Recall the game semantics of intuitionistic propositional logic: $\models_{\mathrm{dialog}} \varphi$.

- $\models_{\mathrm{dialog}} \mathrm{p} \rightarrow \neg\neg\mathrm{p}$,
- $\not\models_{\mathrm{dialog}} \neg\neg\mathrm{p} \rightarrow \mathrm{p}$,
- $\not\models_{\mathrm{dialog}} \varphi \vee \neg\varphi$.

Kripke translation (1965) of intuitionistic propositional logic into modal logic:

$$\begin{array}{rcl} \mathrm{K}(\mathrm{p}_i) & := & \Box\mathrm{p}_i \\ \mathrm{K}(\varphi \vee \psi) & := & \mathrm{K}(\varphi) \vee \mathrm{K}(\psi) \\ \mathrm{K}(\neg\varphi) & := & \Box\neg\mathrm{K}(\varphi) \end{array}$$

Intuitionistic Logic (2).

Theorem.

$$\models_{\text{dialog}} \varphi \leftrightarrow \mathbf{S4} \vdash \mathrm{K}(\varphi).$$

Consequently, φ is intuitionistically valid if and only if $\mathrm{K}(\varphi)$ holds on all transitive and reflexive frames.

$$\begin{array}{rcl}
\models_{\text{dialog}} \mathrm{p} \rightarrow \neg\neg\mathrm{p} & \rightsquigarrow & \Box\mathrm{p} \rightarrow \Box\Diamond\Box\mathrm{p} \\
\not\models_{\text{dialog}} \neg\neg\mathrm{p} \rightarrow \mathrm{p} & \rightsquigarrow & \Box\Diamond\Box\mathrm{p} \rightarrow \Box\mathrm{p} \\
\not\models_{\text{dialog}} \varphi \vee \neg\varphi & \rightsquigarrow & \mathrm{K}(\varphi) \vee \Box\neg\mathrm{K}(\varphi) \\
 & & \Box\mathrm{p} \vee \Box\neg\Box\mathrm{p} \\
 & & \Box\mathrm{p} \vee \Box\Diamond\neg\mathrm{p}
\end{array}$$

Provability Logic (1).

Leon Henkin (1952). “If φ is provably equivalent to $\mathsf{PA} \vdash \varphi$, what do we know about φ?”

M. H. Löb, Solution of a problem of Leon Henkin, **Journal of Symbolic Logic** 20 (1955), p.115-118:

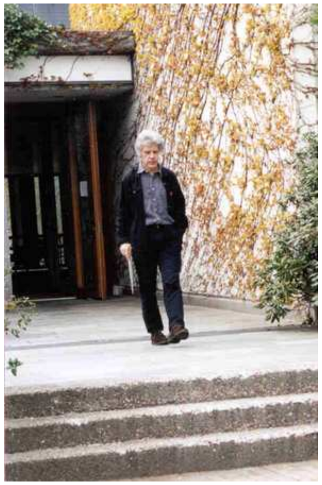
$\mathsf{PA} \vdash ((\mathsf{PA} \vdash \varphi) \rightarrow \varphi)$ implies $\mathsf{PA} \vdash \varphi$.

Interpret $\Box\varphi$ as $\mathsf{PA} \vdash \varphi$. Then Löb's theorem becomes:

$$\text{(Löb) } \Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi.$$

$\mathbf{GL}$ is the modal logic with the axiom (Löb).

Provability Logic (2).



Dick de Jongh

Giovanni Sambin

Theorem (de Jongh-Sambin; 1975). $\mathbf{GL}$ has a fixed-point property.

Corollary. $\mathbf{GL} \vdash \neg\Box\bot \leftrightarrow \neg\Box(\neg\Box\bot)$.

Provability Logic (3).

Theorem (Segerberg-de Jongh-Kripke; 1971). $\mathbf{GL} \vdash \varphi$ if and only if φ is true on all transitive converse wellfounded frames.

A translation R from the language of model logic into the language of arithmetic is called a realization if

$$\begin{aligned} R(\bot) &= \bot \\ R(\neg\varphi) &= \neg R(\varphi) \\ R(\varphi \vee \psi) &= R(\varphi) \vee R(\psi) \\ R(\Box\varphi) &= \mathsf{PA} \vdash R(\varphi). \end{aligned}$$

Theorem (Solovay; 1976). $\mathbf{GL} \vdash \varphi$ if and only if for all realizations R, PA $\vdash R(\varphi)$.

Modal Logics of Models (1).

One example: Modal logic of forcing extensions.

Joel D. Hamkins

A function H is called a Hamkins translation if

$$
\begin{aligned}
H(\bot) &= \bot \\
H(\neg\varphi) &= \neg H(\varphi) \\
H(\varphi \vee \psi) &= H(\varphi) \vee H(\psi) \\
H(\Diamond\varphi) &= \text{“there is a forcing extension in which } H(\varphi) \text{ holds”.}
\end{aligned}
$$

The Modal Logic of Forcing: $\mathbf{Force} := \{\varphi\,;\ \mathsf{ZFC} \vdash H(\varphi)\}$.

Modal Logics of Models (2).

$\mathbf{Force} := \{\varphi\,;\, \mathsf{ZFC} \vdash H(\varphi)\}$.

Theorem (Hamkins).

1. $\mathbf{Force} \nvdash \mathbf{S5}$.
2. $\mathbf{Force} \vdash \mathbf{S4}$.
3. There is a model of set theory $\mathbf{V}$ such that the Hamkins translation of $\mathbf{S5}$ holds in that model.

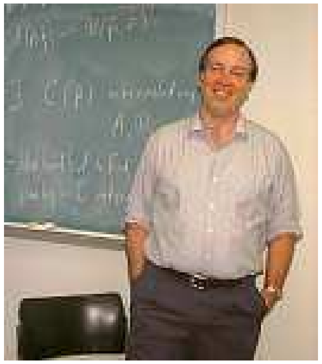
Joel D. **Hamkins**, A simple maximality principle, **Journal of Symbolic Logic** 68 (2003), p. 527–550

Theorem (Hamkins-L). $\mathbf{Force} = \mathbf{S4.2}$.

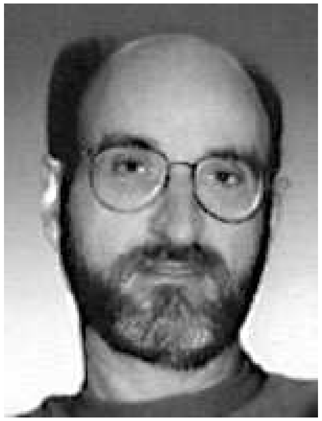
Recent developments.

ASL Annual Meeting 2000 in Urbana-Champaign:

Sam **Buss**, Alekos **Kechris**, Anand **Pillay**, Richard **Shore**, The prospects for mathematical logic in the twenty-first century, **Bulletin of Symbolic Logic** 7 (2001), p.169-196



Sam **Buss**



Alekos **Kechris**

Anand **Pillay**

Richard **Shore**

Proof Theory.

- Generalized Hilbert's Programme (Gentzen-style analysis of proof systems).

Wolfram Pohlers   Gerhard Jäger   Michael Rathjen

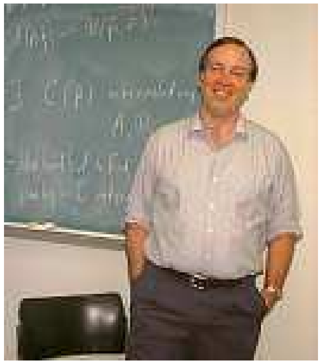
Proof Theory.

- Generalized Hilbert's Programme (Gentzen-style analysis of proof systems).
- Reverse Mathematics.

Harvey Friedman   Steve Simpson

Proof Theory.

- Generalized Hilbert's Programme (Gentzen-style analysis of proof systems).
- Reverse Mathematics.
- Bounded Arithmetic.



Sam Buss Arnold Beckmann

Reverse Mathematics.

“The five systems of reverse mathematics”

- RCA_0 “recursive comprehension axiom”.
- ACA_0 “arithmetic comprehension axiom”.
- WKL_0 “weak König’s lemma”.
- ATR_0 “arithmetic transfinite recursion”.
- Π^1_1-CA_0 “Π^1_1-comprehension axiom”.

Empirical Fact. Almost all theorems of classical mathematics are equivalent to one of the five systems.

Stephen G. **Simpson**, Subsystems of second order arithmetic, Springer-Verlag, Berlin 1999 [*Perspectives in Mathematical Logic*]

Recursion Theory.

- Investigate the structure of the Turing degrees. $\mathcal{D} := \langle \wp(\mathbb{N})/{\equiv_{\mathrm{T}}}, \leq_{\mathrm{T}} \rangle$.
- **Question.** Is $\mathcal{D}$ rigid, *i.e.*, is there a nontrivial automorphism of $\mathcal{D}$?
- **Theorem** (Slaman-Woodin). For any automorphism π of $\mathcal{D}$ and any $\mathbf{d} \geq \mathbf{0}''$, we have $\pi(\mathbf{d}) = \mathbf{d}$.
- **Corollary.** There are at most countably many different automorphisms of $\mathcal{D}$.
- Other degree structures (*e.g.*, truth-table degrees).
- Connections to randomness and Kolmogorov complexity.
- Computable Model Theory.

Model Theory (1).

Theorem (Morley). Every theory that is κ-categorical for one uncountable κ is κ-categorical for all uncountable κ.

Michael Morley

$\rightsquigarrow$ **Stability Theory**
(Baldwin, Lachlan, Shelah)

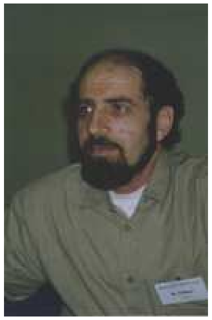
Saharon Shelah

"**Few is beautiful!**"
$\rightsquigarrow$ Classification Theory

Development of new forcing techniques (proper forcing)

Model Theory (2).

- **Geometric Model Theory.**



Boris Zil'ber

Greg Cherlin

Ehud Hrushovski

Applications to algebraic geometry: Geometric Mordell-Lang conjecture.

- **o-Minimality.**

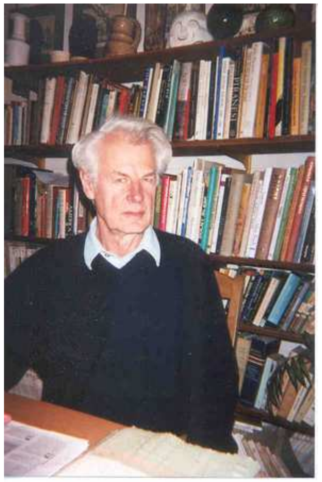
Lou van den Dries

Anand Pillay

Julia Knight

Set Theory.

- **Combinatorial Set Theory**: applications in analysis and topology; using forcing ("Polish set theory").
- **Large Cardinal Theory**: inner model technique.
- **Determinacy Theory**: infinite games and their determinacy; applications to the structure theory of the reals.



Jan Mycielski

Yiannis Moschovakis

Tony (Donald A.) Martin

The Continuum Problem.

Is the independence of CH from the Zermelo-Fraenkel axioms a solution of Hilbert's first problem?

(**Reminder:** Gödel's programme to find new axioms that imply or refute CH.)

- *Shelah's answer*: The question was wrong. The right question should be about other combinatorial objects. There we can prove the "revised GCH" (Sh460). PCF Theory.

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Matt Foreman

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- *Foreman's answer*: Large cardinals can't help, but "generic large cardinals" might.
- *Woodin's answer*: Instead of looking at the statements of new axioms, look at the metamathematical properties of axiom candidates. There is an asymmetry between axioms that imply CH and those that imply $\neg$CH. Woodin's Ω-conjecture.