UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Axiomatic Set Theory

(Axiomatische Verzamelingentheorie)

2003/2004; 2nd Trimester
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Homework Set # 3 Deadline: Thursday, January 29th, 2004

Exercise 3.1 (Zermelo universes are infinite).

Prove (informally) that no structure that satisfies $(\mathbf{I})$ and $(\mathbf{II})$ (extensionality, pairing and emptyset) can be finite. *(Note that there is no need to use* $(\mathbf{VI})$ *(infinity).)*

Hint. Do this by taking an arbitrary structure satisfying $(\mathbf{I})$ and $(\mathbf{II})$ and defining an injection from the natural numbers into it.

Exercise 3.2 (No maximal elements).

(i) If $\langle S, \in \rangle$ is a structure with a binary relation (directed graph), we call $s \in S$ **maximal** if there is no $u \in S$ such that $s \in u$. Show that no structure satisfying $(\mathbf{IIb})$ (pairing) can have a maximal element.

(ii) We call $s \in S$ **almost maximal** if for every $u \in S$, we have

$$s \in u \to s = u.$$

Show that if $\langle S, \in \rangle$ is a structure satisfying $(\mathbf{I})$ and $(\mathbf{IIb})$ (extensionality and pairing), and s is an almost maximal element, then $s = \{s\}$. Also give an example of a finite structure with an almost maximal element satisfying $(\mathbf{I})$ and $(\mathbf{IIb})$.

Hint. Note that the structure shouldn't satisfy $(\mathbf{IIa})$ because otherwise you can't get a finite example by **Exercise 3.1**.

(iii) Show that no structure satisfying $(\mathbf{I}) + (\mathbf{II}) + (\mathbf{III})$ (extensionality, pairing, union and separation) can have an almost maximal element.

Hint. Use the fact that we proved $\forall x \exists y (y \notin x)$ from $(\mathbf{I}) + (\mathbf{II}) + (\mathbf{III})$ in the lecture.

Exercise 3.3 (Preparations for cardinal arithmetic).

Work in the theory $(\mathbf{I}) + (\mathbf{II}) + (\mathbf{III}) + (\mathbf{IV}) + (\mathbf{V})$. Assume that $A =_{\mathrm{c}} A'$ and $B =_{\mathrm{c}} B'$. Prove that

$$A \uplus B =_{\mathrm{c}} A' \uplus B', A \times B =_{\mathrm{c}} A' \times B', (A \to B) =_{\mathrm{c}} (A' \to B').$$

*(This is **Exercise 4.18** in Moschovakis' book (p. 40).)*