Universiteit van Amsterdam
Institute for Logic, Language and Computation

Advanced Topics in Set Theory

2003/2004; 1st Semester
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Homework Set # 6. Deadline: November 25th, 2003

Exercise 20 ("All sets are Borel") *[Repeated from Homework Set # 5].*
Assume that $\mathbb{N}^{\mathbb{N}}$ is the countable union of countable sets.
Show that every set of reals is $\boldsymbol{\Delta}^0_4$. Why is this not in conflict with the construction of a $\boldsymbol{\Pi}^0_4$-universal set?

Remark. Note that "For every countable family $\langle X_i\,;\, i \in \mathbb{N}\rangle$ of countable sets there is a family $\langle x_{ij}\,;\, i,j \in \mathbb{N}\rangle$ listing all elements of $\bigcup_{i\in I} X_i$" is a particular instance of the Axiom of Choice. If the reals are a countable union of countable sets, this statement must be false. (Thus, our assumption is a very blatant violation of the Axiom of Choice – be extremely careful to avoid any use of the Axiom of Choice in this exercise.)
It is a result of Feferman and Lévy that "$\mathbb{N}^{\mathbb{N}}$ is the countable union of countable sets" is consistent with ZF.

Exercise 21. (Marczewski-Burstin algebras).
Recall that for a set $\mathcal{A} \subseteq \wp(\mathbb{N}^{\mathbb{N}})$ the **Marczewski-Burstin algebra** $\mathbf{MB}(\mathcal{A})$ is defined by

$$\mathbf{MB}(\mathcal{A}) := \{S\,;\, \forall A \in \mathcal{A}\, \exists A^* \in \mathcal{A}\, (A^* \subseteq A \,\&\, (A^* \subseteq S \,\vee\, A^* \cap S = \varnothing))\,\}.$$

Let $\mathcal{B}$ be the standard basis of the topology of the Baire space. Show that $\mathbf{MB}(\mathcal{B})$ is not a σ-algebra. In particular, there is a Borel set which is not in $\mathbf{MB}(\mathcal{B})$.

Exercise 22. (The Marczewski ideal).
Let $\mathcal{P}$ be the set of perfect subsets of the Baire space. Recall that we say that we defined the Marczewski ideal $(s^0) = \mathbf{MB}^0(\mathcal{P})$ by

$$\mathbf{MB}^0(\mathcal{P}) := \{S\,;\, \forall A \in \mathcal{A}\, \exists A^* \in \mathcal{A}\, (A^* \subseteq A \,\&\, (A^* \cap S = \varnothing))\,\}.$$

Prove that the Marczewski ideal is a σ-ideal, *i.e.*, countable unions of sets in the ideal are again in the ideal.

Hint. Think of the perfect sets as trees and construct what is called a **fusion sequence of trees**: keep splittings while thinning out the tree below the splitting nodes.