Universiteit van Amsterdam
Institute for Logic, Language and Computation

Advanced Topics in Set Theory

2003/2004; 1st Semester
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Homework Set # 4. Deadline: November 11th, 2003

Exercise 14 (Lifting measures to ultrapowers).

This exercise is related to **Exercise 13**. We write $\kappa \to (\text{club})^\kappa$ for the **strong club partition property**: If $\chi : [\kappa]^\kappa \to 2$ is a partition then there is a club set $C \in \mathcal{C}_\kappa$ such that χ is constant on $[C]^\kappa$. The strong club partition property is a strengthening of the strong partition relation $\kappa \to (\kappa)^\kappa$ and violates the Axiom of Choice. So make sure that you don't use it in this exercise.
Assume that κ has the strong club partition property, that U is an ultrafilter on κ, and that $\mathrm{Ult}(\kappa, U)$ is wellfounded. Then $\mathrm{Ult}(\kappa, U)$ is isomorphic to an ordinal $\lambda > \kappa$. Prove that $\mathbf{lift}(U)$, called the **lift of** U **by itself**, and defined by

$$A \in \mathbf{lift}(U) \iff \exists C \in \mathcal{C}_\kappa \, \forall f \in C^\kappa \; ([f]_U \in A)$$

is an ultrafilter on λ.

Exercise 15 (Round brackets and square brackets).

Let $\lambda < \kappa$, $\mu \leq \kappa$. Prove that $\kappa \to (\mu)^{<\omega}_\lambda$ implies $\kappa \to [\mu]^{<\omega}_{\lambda,<\omega_1}$.

Exercise 16 (Ramsey cardinals and infinitary Chang Conjectures).

Let $\mathcal{L}_\lambda$ be the language of set theory augmented with a distinguished predicate symbol $\dot{\mathsf{P}}$ and λ many constant symbols $\langle \dot{\mathsf{c}}_\alpha \, ; \, \alpha < \lambda \rangle$. By **the** $\mathcal{L}_\lambda$ **version of** $\langle \kappa, \xi \rangle \twoheadrightarrow \langle \mu, < \nu \rangle$ we mean the statement

> "every $\mathcal{L}_\lambda$ structure $\langle X, Y, \langle x_\alpha \, ; \, \alpha < \lambda \rangle \rangle$ with $\mathrm{Card}(X) = \kappa$ and $\mathrm{Card}(Y) = \xi$ has an elementary substructure $\langle X^*, Y^*, \langle x_\alpha \, ; \, \alpha < \lambda \rangle \rangle$ such that $\mathrm{Card}(X^*) = \mu$ and $\mathrm{Card}(Y^*) < \nu$".

Prove that if κ is Ramsey, then for all $\lambda < \kappa$, the $\mathcal{L}_\lambda$ version of $\langle \kappa, \lambda^+ \rangle \twoheadrightarrow \langle \kappa, < \lambda^+ \rangle$ holds.

Hint. You may use without proof that $\kappa \to (\kappa)^{<\omega}_\lambda$ for all $\lambda < \kappa$ (Kanamori, Proposition 7.14 (c)). Develop a notion of Skolem function for the infinitary language $\mathcal{L}_\lambda$ and follow the idea of Rowbottom's Skolemization Theorem with **Exercise 15** in mind. Note that this exercise is close to Kanamori, Theorem 8.4.

Exercise 17 (Measurable cardinals and the size of $\mathbf{L}$).

Assume that there is a proper class of measurable cardinals. Then every regular cardinal in $\mathbf{V}$ is inaccessible in $\mathbf{L}$.

Hint. Use **Exercise 16** and the proof idea of "If there is a Rowbottom cardinal, then $\omega_1^{\mathbf{L}}$ is countable" to show that if κ is Ramsey and $\lambda < \kappa$, then $\mathrm{Card}(\wp^{\mathbf{L}}(\lambda)) = \mathrm{Card}(\lambda)$. Compare Kanamori, Corollary 8.5.